

*Citation for published version:*

Katzourakis, N & Pryer, T 2020, 'Second-Order  $L$  Variational Problems and the  $-$ Polylaplacian', *Advances in Calculus of Variations*, vol. 13, no. 2, pp. 115-140. <https://doi.org/10.1515/acv-2016-0052>

*DOI:*

[10.1515/acv-2016-0052](https://doi.org/10.1515/acv-2016-0052)

*Publication date:*

2020

*Document Version*

Peer reviewed version

[Link to publication](https://doi.org/10.1515/acv-2016-0052)

**University of Bath**

**Alternative formats**

If you require this document in an alternative format, please contact:  
[openaccess@bath.ac.uk](mailto:openaccess@bath.ac.uk)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

# SECOND ORDER $L^\infty$ VARIATIONAL PROBLEMS AND THE $\infty$ -POLYLAPLACIAN

NIKOS KATZOURAKIS AND TRISTAN PRYER

**ABSTRACT.** In this paper we initiate the study of 2nd order variational problems in  $L^\infty$ , seeking to minimise the  $L^\infty$  norm of a function of the hessian. We also derive and study the respective PDE arising as the analogue of the Euler-Lagrange equation. Given  $H \in C^1(\mathbb{R}_s^{n \times n})$ , for the functional

$$(1) \quad E_\infty(u, \mathcal{O}) = \|H(D^2u)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{2,\infty}(\Omega), \quad \mathcal{O} \subseteq \Omega,$$

the associated equation is the fully nonlinear 3rd order PDE

$$(2) \quad A_\infty^2 u := (H_X(D^2u))^{\otimes 3} : (D^3u)^{\otimes 2} = 0.$$

Special cases arise when  $H$  is the Euclidean length of either the full hessian or of the Laplacian, leading to the  $\infty$ -Polylaplacian and the  $\infty$ -Bilaplacian respectively. We establish several results for (1) and (2), including existence of minimisers, of absolute minimisers and of “critical point” generalised solutions, proving also variational characterisations and uniqueness. We also construct explicit generalised solutions and perform numerical experiments.

## CONTENTS

1. Introduction	1
2. A quick guide of Young measures valued into spheres	7
3. Derivation of the $L^\infty$ equation from $L^p$ as $p \rightarrow \infty$	8
4. Characterisation of $A_\infty^2$ via the flow map of an ODE	9
5. Existence of 2nd order Minimisers & Absolute Minimisers	9
6. Variational characterisation of $A_\infty^2$ via $AM^2$ & uniqueness	17
7. Existence of $\mathcal{D}$ -solutions to the Dirichlet problem for $A_\infty^2$	22
8. Explicit $p$ -Biharmonic and $\infty$ -Biharmonic functions in $1D$	25
9. Numerical approximations of $\infty$ -Biharmonic functions	27
References	29

## 1. INTRODUCTION

In this paper we initiate the study of higher order variational problems in the space  $L^\infty$  and of their respective associated equations. As a first step, we consider the problem of minimising the  $L^\infty$  norm of a function of the hessian and study its connection to a respective PDE arising as the analogue of the Euler-Lagrange

---

*Key words and phrases.* Calculus of Variations in  $L^\infty$ ;  $\infty$ -Polylaplacian;  $\infty$ -Bilaplacian; Generalised solutions; Fully nonlinear equations; Young measures; Baire Category method; Convex Integration.

N.K. has been partially financially supported by the EPSRC grant EP/N017412/1.

T.P. has been partially financially supported by the EPSRC grant EP/P000835/11.

equation. More precisely, let  $\Omega \subseteq \mathbb{R}^n$  be a bounded open set,  $n \in \mathbb{N}$ . For a real function  $u \in C^3(\Omega)$ , the gradient, the hessian and the 3rd order derivative are denoted respectively by

$$\begin{aligned} Du &= (D_i u)_{i=1}^n : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\ D^2 u &= (D_{ij}^2 u)_{i,j=1}^n : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_s^{n^{\otimes 2}}, \\ D^3 u &= (D_{ijk}^3 u)_{i,j,k=1}^n : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_s^{n^{\otimes 3}}, \end{aligned}$$

and in general for any  $N \in \mathbb{N}$  the  $N$ th order derivative  $D^N u$  is valued in the space  $\mathbb{R}_s^{n^{\otimes N}}$ , the  $N$ -fold symmetric tensor power of  $\mathbb{R}^n$ :

$$\mathbb{R}_s^{n^{\otimes N}} := \left\{ T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n \equiv \mathbb{R}^{n^{\otimes N}} \mid T_{i_1 \dots i_N} = T_{\sigma(i_1 \dots i_N)}, \sigma \text{ permutation} \right\}.$$

Given a fixed function  $H \in C^1(\mathbb{R}_s^{n^{\otimes 2}})$ , we consider the supremal functional

$$(1.1) \quad E_\infty(u, \mathcal{O}) := \|H(D^2 u)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{2,\infty}(\Omega), \quad \mathcal{O} \subseteq \Omega \text{ measurable}.$$

It turns out that the associated PDE which plays the role of ‘‘Euler-Lagrange’’ equation for (1.1) is the following *fully nonlinear PDE of 3rd order*:

$$(1.2) \quad A_\infty^2 u := (H_X(D^2 u))^{\otimes 3} : (D^3 u)^{\otimes 2} = 0.$$

The notation  $(\cdot)^{\otimes N}$  symbolises the  $N$ -fold tensor power of the object in the bracket and  $H_X$  denotes the gradient of  $H$  with respect to its matrix argument, whilst ‘‘:’’ is a higher order contraction which extends the usual Euclidean inner product of the space of matrices. In index form, (1.2) reads

$$\sum_{i,j,k,l,p,q=1}^n \left( H_{X_{ij}}(D^2 u) H_{X_{kl}}(D^2 u) H_{X_{pq}}(D^2 u) \right) D_{ikl}^3 u D_{jpq}^3 u = 0.$$

Further, by contracting derivatives we may rewrite (1.2) as

$$(1.3) \quad A_\infty^2 u = H_X(D^2 u) : D(H(D^2 u)) \otimes D(H(D^2 u)) = 0.$$

Special cases of the PDE (1.2) arise when  $H$  is either the Euclidean length of the hessian,  $D^2 u$ , in  $\mathbb{R}_s^{n^{\otimes 2}}$  (squared)  $|D^2 u|^2 = \sum_{i,j=1}^n (D_{ij}^2 u)^2 = D^2 u : D^2 u$  or the absolute value of the Laplacian,  $\Delta u$  (squared), giving rise to the following respective PDEs which (borrowing the terminology from Differential Geometry, see e.g. [M, HM, GS]) we call the  $\infty$ -Polylaplacian and the  $\infty$ -Bilaplacian:

$$(1.4) \quad \Pi_\infty^2 u := (D^2 u)^{\otimes 3} : (D^3 u)^{\otimes 2} = 0,$$

$$(1.5) \quad \Delta_\infty^2 u := (\Delta u \mathbf{I})^{\otimes 3} : (D^3 u)^{\otimes 2} = 0.$$

In (1.5), ‘‘ $\mathbf{I}$ ’’ is the identity matrix. An equivalent way to write  $\Pi_\infty^2$  and  $\Delta_\infty^2$  (after a rescaling) is respectively

$$D^2 u : D(|D^2 u|^2) \otimes D(|D^2 u|^2) = 0, \quad (\Delta u)^3 |D(\Delta u)|^2 = 0.$$

The study of 1st order variational problems when minimising a function of the gradient

$$(1.6) \quad E_\infty(u, \mathcal{O}) := \|H(Du)\|_{L^\infty(\mathcal{O})}, \quad u \in W^{1,\infty}(\Omega), \quad \mathcal{O} \subseteq \Omega \text{ measurable},$$

is by now quite standard and has been pioneered by Aronsson who first considered (1.6) in the 1960s ([A1]-[A7]). In this case, the respective PDE is quasilinear of 2nd order and is commonly known as the “Aronsson equation”:

$$(1.7) \quad A_\infty u := H_p(Du) \otimes H_p(Du) : D^2 u = 0.$$

An important special case of (1.7) arises for  $H(p) = |p|^2$  and is known as the  $\infty$ -Laplacian:  $\Delta_\infty u := Du \otimes Du : D^2 u = 0$ . The field has undergone a marvellous development since then, especially in the 1990s when the advent of the theory of Viscosity Solutions for fully nonlinear 2nd order PDEs made possible the rigorous study of the non-divergence equation (1.7) and of its non-smooth solutions (for a pedagogical introduction with numerous references we refer to [K7, C]). The popularity of this area owes to both the intrinsic mathematical interest as well as to the importance for applications, since minimisation of the maximum “energy” provides more realistic models than the standard integral counterparts of average “energy”. Let us also note that the vectorial first order case is under active research and since the early 2010s is being developed very rapidly. To the best of our knowledge, the systematic study has been initiated by the first author (see [K1]-[K6], [K8]-[K11], and the joint contributions with Abugirda, Ayanbayev, Croce and Pisante [AK, 2, 1, CKP]), while the second author has been working on the numerical analysis of them ([KP, P, LP1, LP2]).

As it is well known from the 1st order case of (1.6), supremal functionals lack “locality” and the requirement of minimality has to be imposed at the outset on all subdomains, not just the domain itself as in the case of integral functionals. In particular, mere minimisers of (1.1) are not truly optimal and may not solve in any sense the PDE (1.2). The variational principle we will be considering for (1.1) is the following extension of Aronsson’s notion of Absolute Minimisers:

**Definition 1.** A function  $u \in W^{2,\infty}(\Omega)$  is called a *2nd order Absolute Minimiser* of (1.1) when

$$E_\infty(u, \Omega') \leq E_\infty(u + \phi, \Omega'), \quad \forall \Omega' \Subset \Omega, \quad \forall \phi \in W_0^{2,\infty}(\Omega').$$

The set of 2nd order Absolute Minimisers will be symbolised by  $AM^2(E_\infty, \Omega)$ .

We would like to emphasise that, as the explorative results in this paper will make apparent,

*the higher order (scalar) case of (1.1)-(1.2) can not be developed “by analogy” to the first order case (neither scalar nor vectorial) and unexpected phenomena arise.*

For example, a fundamental difficulty associated to (1.1) is that *even the 1-dimensional problem of minimising  $|u''|^2$  in  $L^\infty$  (or in  $L^p$ ) is not trivial; in particular, even in this case the minimisers are non-polynomial and actually have singular points, being non- $C^2$  and just  $W^{2,\infty}$  (let alone  $C^3$ )*. An extra difficulty associated to (1.5) is that the respective functional  $(u, A) \mapsto \|(\Delta u)^2\|_{L^\infty(A)}$  is not coercive in  $W^{2,\infty}(\Omega)$  but instead in the space

$$\left\{ u \in \bigcap_{1 < p < \infty} W^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega) \right\}$$

because of the failure of the Calderon-Zygmund  $L^p$ -estimates in the extreme case  $p = \infty$  (see e.g. [GM, GT]). More importantly and even more unexpectedly, the relevant PDE (1.2) is not any more 2nd order quasilinear and degenerate elliptic, but

instead 3rd order fully nonlinear since it is quadratic in the highest order derivative; moreover, it is highly degenerate but in no obvious fashion elliptic. To the best of our knowledge this is the first instance in Calculus of Variations in general where a fully nonlinear PDE of odd order is actually variational and in addition not a null Lagrangian. Even the 1-dimensional version of the  $\infty$ -Polylaplacian/ $\infty$ -Bilaplacian is not trivial; in fact, for  $n = 1$  both equations (1.4)-(1.5) simplify to

$$(1.8) \quad \Delta_\infty^2 u = (u'')^3 (u''')^2 = 0$$

and it is easy to see that its solutions can not in general be  $C^3$ . Accordingly,

$$\begin{cases} (u'')^3 (u''')^2 = 0, & \text{in } \Omega, \\ u = g, u' = g', & \text{on } \partial\Omega \end{cases}$$

is solvable in the class of  $C^3$  solutions if and only if  $g$  is a quadratic polynomial. In general, even the “best candidate” solution coming from the limit of the  $p$ -Bilaplacian as  $p \rightarrow \infty$  is not  $C^2$  and  $u''$  is discontinuous.

Perhaps the greatest difficulty associated to the study of (1.2) is that all standard approaches in order to define generalised solutions based on maximum principle or on integration-by-part considerations seem to fail. As highlighted above, there is a real necessity for such a notion for (1.2) even when  $n = 1$ . More concretely, by a separation of variables of the form  $u(x, y) = f(x) + g(y)$  on  $\mathbb{R}^2$ , one easily arrives as Aronsson did in [A6] to the singular global  $\infty$ -Polyharmonic function on  $\mathbb{R}^2$   $u(x, y) = |x|^{\frac{12}{5}} - |y|^{\frac{12}{5}}$  which is saddle-shaped but not thrice differentiable on the axes because  $|D^2 u(x, y)|^2 \cong |x|^{4/5} + |y|^{4/5}$ . Further singular solutions without 3rd order derivatives arise by the special class of solutions to the fully nonlinear 2nd order equation  $H(D^2 u) = c$ .

Motivated in part by the systems arising in vectorial Calculus of Variations in  $L^\infty$ , the first author has recently introduced in [K8, K9] a new efficient theory of generalised solutions which applies to fully nonlinear systems of any order

$$\mathcal{F}(\cdot, u, Du, D^2 u, \dots, D^N u) = 0, \quad \text{in } \Omega,$$

and allows for *merely measurable* mappings as solutions. This general approach of the so-called  $\mathcal{D}$ -solutions is based on the probabilistic representation of those derivatives which do not exist classically. The tool in achieving this is the weak\* compactness of difference quotients in the Young measures valued into a compactification of the “space of jets”. For the special case of the 3rd order PDE (1.2), we can motivate the idea as follows: let  $u$  be a  $W^{3,\infty}(\Omega)$  strong solution of

$$(1.9) \quad \mathcal{F}(D^2 u, D^3 u) = 0, \quad \text{a.e. on } \Omega.$$

In order to interpret the 3rd derivative rigorously for just  $W_{\text{loc}}^{2,\infty}(\Omega, \mathbb{R}^N)$  (which is the natural regularity class for (1.2) arising from (1.1)), we argue as follows: let us restate (1.9) as

$$(1.10) \quad \int_{\mathbb{R}_s^{n \otimes 3}} \Phi(\mathbf{X}) \mathcal{F}(D^2 u(x), \mathbf{X}) d[\delta_{D^3 u(x)}](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega,$$

for any  $\Phi \in C_c(\mathbb{R}_s^{n \otimes 3})$  with compact support. Namely, we view the 3rd derivative tensor  $D^3 u$  as a probability-valued mapping  $\Omega \subseteq \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}_s^{n \otimes 3})$  which is given by  $x \mapsto \delta_{D^3 u(x)}$ , the Dirac measure at the 3rd derivative. Also, we may rephrase

that  $D^3u$  is the a.e. sequential limit of the difference quotients  $D^{1,h}D^2u$  of the hessian along infinitesimal sequences  $(h_m)_1^\infty$  by writing

$$(1.11) \quad \delta_{D^{1,h_m}D^2u} \xrightarrow{*} \delta_{D^3u}, \quad \text{as } m \rightarrow \infty.$$

The weak\* convergence in (1.11) is taken in the set of Young measures valued into the tensor space  $\mathbb{R}_s^{n^{\otimes 3}}$  (the set of weakly\* measurable probability-valued maps  $\Omega \subseteq \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}_s^{n^{\otimes 3}})$ , for details see Section 2 and [CFV, FG, V, Pe, FL]). The idea arising from (1.10)-(1.11) is that perhaps general probability-valued “diffuse 3rd derivatives” could arise for twice differentiable maps which may not be the concentrations  $\delta_{D^3u}$ . This is actually possible upon replacing  $\mathbb{R}_s^{n^{\otimes 3}}$  by its 1-point sphere compactification in order to gain some compactness:  $\overline{\mathbb{R}}_s^{n^{\otimes 3}} := \mathbb{R}_s^{n^{\otimes 3}} \cup \{\infty\}$ . Then, the maps  $(\delta_{D^{1,h}D^2u})_{h \neq 0}$  considered as Young measures do have subsequential weak\* limits which can play the role of generalised 3rd order derivatives:

**Definition 2** (Diffuse 3rd derivatives, cf. [K8, K9]). For any  $u \in W_{\text{loc}}^{2,1}(\Omega)$ , we define its diffuse 3rd derivatives  $\mathcal{D}^3u$  as the limits of difference quotients of  $D^2u$  along infinitesimal sequences  $(h_m)_1^\infty$  in the Young measures valued into  $\overline{\mathbb{R}}_s^{n^{\otimes 3}}$ :

$$\delta_{D^{1,h_m}D^2u} \xrightarrow{*} \mathcal{D}^3u, \quad \text{in } \mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n^{\otimes 3}}), \quad \text{as } m \rightarrow \infty.$$

If  $\{e^1, \dots, e^n\}$  stands for the standard basis of  $\mathbb{R}^n$ , then apparently

$$D^{1,h}v := (D_1^{1,h}, \dots, D_n^{1,h}v), \quad D_i^{1,h}v(x) := \frac{1}{h} [v(x + he^i) - v(x)], \quad h \neq 0.$$

Since the set  $\mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n^{\otimes 3}})$  is weakly\* compact, every map possesses at least one diffuse 3rd derivative and actually exactly one if the hessian is a.e. differentiable with measurable derivative (see [K8]).

**Definition 3** (Twice differentiable  $\mathcal{D}$ -solutions of 3rd order PDEs, cf. [K8]). Let  $\mathcal{F} : \mathbb{R}_s^{n^{\otimes 2}} \times \mathbb{R}_s^{n^{\otimes 3}} \rightarrow \mathbb{R}$  be Borel measurable. A function  $u \in W_{\text{loc}}^{2,1}(\Omega, \mathbb{R}^N)$  is a  $\mathcal{D}$ -solution to

$$(1.12) \quad \mathcal{F}(D^2u, D^3u) = 0, \quad \text{in } \Omega,$$

when for any diffuse 3rd derivative  $\mathcal{D}^3u \in \mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n^{\otimes 3}})$  and any  $\Phi \in C_c(\mathbb{R}_s^{n^{\otimes 3}})$ ,

$$\int_{\overline{\mathbb{R}}_s^{n^{\otimes 3}}} \Phi(\mathbf{X}) \mathcal{F}(D^2u(x), \mathbf{X}) d[\mathcal{D}^3u(x)](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega.$$

The notion of generalised solution of Definitions 2 & 3 will be the central notion of solution for our fully nonlinear PDE (1.2). For more on the theory of  $\mathcal{D}$ -solutions for general systems, analytic properties, existence/uniqueness/partial regularity results see [K8]-[K11] and [2, 1, AK, CKP].

We note that the interpretation of (1.2) in the “contracted” form (1.3) is not generally appropriate for non- $C^3$  solutions; interpreting the “expanded” equation (1.2) in a weak sense is essential. In particular, even in the 1D case of (1.8), the results of Sections 8-9 demonstrate that seeing the  $\infty$ -Poly/Bilaplacian as  $u''((|u''|^2)')^2$  is not appropriate even when  $n = 1$  since there exist solutions for which  $u''$  is piecewise constant and hence the distributional derivative  $(|u''|^2)'$  is a measure (whose square is not well defined!).

In this paper we are concerned with the study of 2nd order Absolute Minimisers of (1.1), of  $\mathcal{D}$ -solutions to (1.2) and of their analytic properties and their connection.

To this end, we prove several things and the table of contents is relatively self-explanatory of the results we obtain in this paper. Below we give a quick description and main highlights:

In Section 2 we give a quick review of the very few ingredients of Young measures into spheres which are utilised in this paper for the convenience of the reader.

In Section 3 we formally derive the equation (1.1) in the limit of the Euler-Lagrange equation of the respective  $L^p$  functional  $u \mapsto \|H(D^2u)\|_{L^p(\Omega)}$  as  $p \rightarrow \infty$ .

In Section 4 we characterise  $C^3$  solutions to (1.2) via the flow map of an ODE system along the orbits of which the energy is constant (Proposition 4).

In Section 5 we prove existence of minimisers for (1.1) given Dirichlet boundary condition on a bounded open set under two sets of weak hypotheses which include both the  $\infty$ -Polylaplacian (1.4) and the  $\infty$ -Bilaplacian (1.5) (Theorems 5, 6). We also give a complete solution to the problem of existence-uniqueness and description of the fine structure of 2nd order Absolute Minimisers of (1.1) and of the corresponding minimising  $\mathcal{D}$ -solutions to (1.2) when  $n = 1$  (Theorem 7).

In Section 6 we establish the necessity of the PDE (1.2) for 2nd order Absolute Minimisers of (1.1) in the class of  $C^3$  solutions (Theorem 14(I)). This is nontrivial even for  $C^3$  solutions because standard 1st order arguments fail to construct test functions in  $W_0^{2,\infty}(\Omega)$  and a deep tool is required, the Whitney extension theorem ([W, M, F]). If further  $H$  depends on  $D^2u$  via the projection  $A : D^2u$  along a fixed matrix (e.g. on the Laplacian  $\Delta u = D^2u : I$ ), we prove sufficiency as well and hence equivalence (Theorem 14(II)). As a consequence, in the latter case we deduce uniqueness in the  $C^3$  class for (1.1) and (1.2) (Corollary 15).

In Section 7 we employ the Dacorogna-Marcellini Baire Category method ([DM, D]) which in a sense is the analytic counterpart to Gromov's Convex integration and establish the existence of non-minimising "critical point"  $\mathcal{D}$ -solutions to the Dirichlet problem (1.2) (Theorem 18). We construct  $\mathcal{D}$ -solutions in  $W_g^{2,\infty}(\Omega)$  with the extra geometric property of solving strongly the fully nonlinear equation  $H(D^2u) = c$  at any large enough energy level  $c > 0$ . Interestingly, *we do not assume any kind of convexity or level-convexity or "BJW-convexity"* (i.e. the notion of  $L^\infty$ -quasiconvexity introduced in [BJW2]). This method has previously been applied in the construction of critical point vectorial  $\mathcal{D}$ -solutions in the first order case in [K9, CKP] and has some vague relevance to the method used in [DS] to construct solutions to the Euler equations.

In Section 8 we solve explicitly the  $p$ -Bilaplacian (weakly) and the  $\infty$ -Bilaplacian (in the  $\mathcal{D}$ -sense) in the case  $n = 1$  (Theorem 21). In particular, the  $p$ -Biharmonic functions are  $C^\infty$  except for at most one point in the domain and  $\infty$ -Biharmonic functions are smooth except for at most two points in the domain. In the  $\infty$ -case, the  $\mathcal{D}$ -solutions we construct are non-minimising but have fixed energy level and this allows to have uniqueness (absolutely minimising  $\mathcal{D}$ -solutions are constructed in Theorem 7).

Finally in Section 9 we perform some numerical experiments by considering the solutions of the  $p$ -Bilaplacian for given Dirichlet data and large  $p$ . The experiments confirm numerically that the 2nd derivatives of  $\infty$ -Biharmonic functions generally can not be continuous (this is actually proved in Theorems 7 and 14). Even more interestingly, for  $n = 2$  the Laplacian of  $\infty$ -Biharmonic functions appears to be piecewise constant and we have the emergence of non-trivial interfaces whereon it is discontinuous and actually changes sign when crossing the interface. Further,

for the “balanced” symmetric energy  $H(X) = |X : I|^2$  we are using, the values of Laplacian appear to be opposite on the phases, whilst the absolute value of it seems to extend to a constant function throughout the domain (this is proved in Theorem 7 for  $n = 1$ ).

Although in this paper we do not consider any immediate applications of our results, we would like to point out that 2nd order minimisation problems in  $L^\infty$  are very important in several areas of pure and applied Mathematics. In particular, in the papers [MS] and [S] the authors consider the problem of minimising in  $L^\infty$  the Gaussian curvature (if  $n = 2$ ) and the scalar curvature (if  $n \geq 3$ ) of a fixed background Riemannian manifold over a conformal class of deformations of the metric. Although they consider (mere) minimisers and not absolute minimisers of their geometric functionals which is the appropriate notion in  $L^\infty$ , the method they use to construct them is via  $L^p$  approximations and this seems to select the “good” absolutely minimising object. The theoretical and numerical observations we make in this paper are compatible with phenomena of piecewise constant energy and interfaces of discontinuities in the differential-geometric context of minimisation of the curvature in  $L^\infty$  in [MS, S]. Also, after this work had been completed and appeared as a preprint, we learned that in the paper [AB] Aronsson and Barron had already previously derived our PDE (2) (Remark 4.9 on p. 78) without investigating it further. We conclude by noting that in our companion paper [KP2] we establish rigorous numerical approximations for  $\infty$ -Polyharmonic and  $\infty$ -Biharmonic functions and we also consider concrete applications.

## 2. A QUICK GUIDE OF YOUNG MEASURES VALUED INTO SPHERES

Here we collect some rudiments of Young measures taken from [K8] which can be found in greater generality and different guises e.g. in [CFV, FG]. Let  $\Omega \subseteq \mathbb{R}^n$  be open and let us consider the Banach space of  $L^1$  maps valued in the space of continuous functions over the compact manifold  $\overline{\mathbb{R}}_s^{n \otimes 3}$  (for more see e.g. [Ed, FL]):  $L^1(\Omega, C(\overline{\mathbb{R}}_s^{n \otimes 3}))$ . The elements of this space are Carathéodory functions  $\Phi : \Omega \times \overline{\mathbb{R}}_s^{n \otimes 3} \rightarrow \mathbb{R}$  satisfying

$$\|\Phi\|_{L^1(\Omega, C(\overline{\mathbb{R}}_s^{n \otimes 3}))} = \int_{\Omega} \left\{ \max_{\mathbf{X} \in \overline{\mathbb{R}}_s^{n \otimes 3}} |\Phi(x, \mathbf{X})| \right\} dx < \infty.$$

The dual of this Banach space is given by

$$L_{w^*}^\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}_s^{n \otimes 3})) = L^1(\Omega, C(\overline{\mathbb{R}}_s^{n \otimes 3}))^*$$

and consists of measure-valued maps  $x \mapsto \vartheta(x)$  which are weakly\* measurable, namely the real function  $x \mapsto [\vartheta(x)](\mathcal{U})$  is measurable on  $\Omega$  for any fixed open set  $\mathcal{U} \subseteq \overline{\mathbb{R}}_s^{n \otimes 3}$ . The unit closed ball of this dual space is sequentially weakly\* compact because the  $L^1$  space above is separable. The duality pairing is given by

$$(2.1) \quad \begin{cases} \langle \cdot, \cdot \rangle : L_{w^*}^\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}_s^{n \otimes 3})) \times L^1(\Omega, C(\overline{\mathbb{R}}_s^{n \otimes 3})) \rightarrow \mathbb{R}, \\ \langle \vartheta, \Phi \rangle := \int_{\Omega} \int_{\overline{\mathbb{R}}_s^{n \otimes 3}} \Phi(x, \mathbf{X}) d[\vartheta(x)](\mathbf{X}) dx. \end{cases}$$

**Definition** (Young Measures). *The subset of the unit sphere consisting of probability-valued mappings comprises the Young measures from  $\Omega \subseteq \mathbb{R}^n$  to the sphere  $\overline{\mathbb{R}}_s^{n \otimes 3}$ :*

$$\mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3}) := \left\{ \vartheta \in L_{w^*}^\infty(\Omega, \mathcal{M}(\overline{\mathbb{R}}_s^{n \otimes 3})) : \text{a.e. } x \in \Omega, \vartheta(x) \in \mathcal{P}(\overline{\mathbb{R}}_s^{n \otimes 3}) \right\}.$$



We now record for later use the next standard facts (for the proofs see e.g. [FG]):

- (i) Any measurable map  $U : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_s^{n \otimes 3}$  induces a Young measure  $\delta_U$  given by  $\delta_U(x) := \delta_{U(x)}$ ,  $x \in \Omega$ .
- (ii) The set  $\mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3})$  is sequentially weakly\* compact and convex. In particular, any sequence  $(\delta_{U^m})_1^\infty$  has a subsequence such that  $\delta_{U^m} \xrightarrow{*} \vartheta$  as  $m \rightarrow \infty$ .
- (iii) If  $(U^m)_1^\infty, U^\infty$  are measurable maps  $\Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_s^{n \otimes 3}$ , then  $\delta_{U^m} \xrightarrow{*} \delta_{U^\infty}$  in  $\mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3})$  iff  $U^m \rightarrow U^\infty$  a.e. on  $\Omega$ , after perhaps the passage to subsequences.
- (iv) The following is an one-sided characterisation of weak\* convergence:  $\vartheta^m \xrightarrow{*} \vartheta$  as  $m \rightarrow \infty$  in  $\mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3})$  iff  $\langle \vartheta, \Psi \rangle \leq \liminf_{m \rightarrow \infty} \langle \vartheta^m, \Psi \rangle$  for any function  $\Psi : \Omega \times \overline{\mathbb{R}}_s^{n \otimes 3} \rightarrow (-\infty, +\infty]$  bounded from below, measurable in  $x$  for all  $\mathbf{X}$  and lower semicontinuous in  $\mathbf{X}$  for a.e.  $x$ .

### 3. DERIVATION OF THE $L^\infty$ EQUATION FROM $L^p$ AS $p \rightarrow \infty$

In this section we *formally derive* the equation (1.2) in the limit of the Euler-Lagrange equations of the respective  $L^p$  functionals

$$(3.1) \quad E_p(u, \Omega) := \left( \int_{\Omega} H(D^2 u(x))^p dx \right)^{1/p}$$

as  $p \rightarrow \infty$ . Here the bar denotes average. The idea of approximating an  $L^\infty$  variational problem by  $L^p$  problems is quite standard by now for 1st order problems in both the scalar and the vectorial case (see e.g. [C, K7, K9, P, KP]) and has borne substantial fruit. Heuristically, this expectation stems from the fact that for a fixed function  $u \in W^{2,\infty}(\Omega)$ , we have  $E_p(u, \Omega) \rightarrow E_\infty(u, \Omega)$  as  $p \rightarrow \infty$ . For (3.1), the Euler-Lagrange equation is the 4th order divergence structure PDE

$$(3.2) \quad D^2 : \left( H(D^2 u)^{p-1} H_X(D^2 u) \right) = 0$$

which in index form reads  $\sum_{i,j=1}^n D_{ij}^2 (H(D^2 u)^{p-1} H_{X_{ij}}(D^2 u)) = 0$ . By distributing derivatives and rescaling, a calculation gives

$$\begin{aligned} & \sum_{i,j,k,l,p,q} \left( H_{X_{ij}}(D^2 u) H_{X_{kl}}(D^2 u) H_{X_{pq}}(D^2 u) \right) D_{ikl}^3 u D_{jpq}^3 u \\ &= - \frac{H(D^2 u)}{p-2} \left\{ \sum_{i,j,k,l} \left( D_i (H_{X_{kl}}(D^2 u)) D_{jkl}^3 H_{X_{ij}}(D^2 u) \right. \right. \\ & \quad + D_i (H(D^2 u)) D_j (H_{X_{ij}}(D^2 u)) + D_j (H(D^2 u)) D_j (H_{X_{ij}}(D^2 u)) \\ & \quad \left. \left. + H_{X_{ij}}(D^2 u) H_{X_{kl}}(D^2 u) D_{ijkl}^4 u \right) \right\} \\ & \quad - \frac{H(D^2 u)^2}{(p-1)(p-2)} \left\{ \sum_{i,j,k,l,p,q} \left( H_{X_{ij}X_{kl}X_{pq}}(D^2 u) D_{ipq}^3 u D_{jkl}^3 u \right. \right. \\ & \quad \left. \left. + H_{X_{ij}X_{kl}}(D^2 u) D_{ijkl}^4 u \right) \right\}. \end{aligned}$$

Hence, we obtain  $A_\infty^2 u = (H_X(D^2 u))^{\otimes 3} : (D^3 u)^{\otimes 2} = o(1)$  as  $p \rightarrow \infty$  and hence we obtain (1.2). Note that the Euler-Lagrange equation of (3.1) is 4th order and

quasilinear, while the limiting equation is a highly degenerate 3rd order *fully non-linear* equation. In the next sections we utilise this device of  $L^p$  approximations and prove rigorously the existence of minimisers and absolute minimisers.

#### 4. CHARACTERISATION OF $A_\infty^2$ VIA THE FLOW MAP OF AN ODE

In this brief section, inspired by the 1st order case (see [C, K7, K1]) we give a description of classical solutions to our fully nonlinear PDE (1.1) in terms of the flow of a certain ODE system. In the 1st order case the relevant ODE is a gradient system but in the present case it is more complicated and involves 3rd order derivatives.

**Proposition 4.** *Let  $H \in C^1(\mathbb{R}_s^{n \otimes 2})$ ,  $\Omega \subseteq \mathbb{R}^n$  an open set and  $u \in C^3(\Omega)$ . Consider the continuous vector field*

$$(4.1) \quad \mathcal{V} := H_X(D^2u)D(H(D^2u)) : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

*and the initial value problem*

$$(4.2) \quad \begin{cases} \dot{\gamma}(t) = \text{sgn}(\mathcal{V}(\gamma(t))), & t \neq 0, \\ \gamma(0) = x, & t = 0, \end{cases}$$

*where the initial condition is noncritical, i.e.  $\mathcal{V}(x) \neq 0$  and “sgn” symbolises the sign. Then, along the trajectory we have the differential identity*

$$(4.3) \quad |\mathcal{V}(\gamma(t))| \frac{d}{dt} (H(D^2u(\gamma(t)))) = A_\infty^2 u(\gamma(t))$$

*and hence*

$$A_\infty^2 u = 0 \text{ in } \Omega \iff \begin{cases} \forall x \in \Omega : \mathcal{V}(x) \neq 0, \text{ there is a } C^1 \\ \text{solution } \gamma : (-\varepsilon, \varepsilon) \longrightarrow \Omega \text{ of (4.2):} \\ H(D^2u(\gamma(t))) \equiv H(D^2u(x)), |t| < \varepsilon. \end{cases}$$

Note that, unlike the counterpart 1st order case, the solution of initial value problem (4.2) may not be unique in general.

**Proof of Proposition 4.** In order to conclude it suffices to establish (4.3) the proof of which is a straightforward calculation. The proposition ensues.  $\square$

Heuristically, the meaning of this result is the following: in view of (1.2) the functions with  $H(D^2u) \equiv c$  are special solutions to the PDE. Conversely, all solutions satisfy  $H(D^2u) \equiv c$  at least locally along the trajectories of the 1st order ODE system (4.1)-(4.2).

#### 5. EXISTENCE OF 2ND ORDER MINIMISERS & ABSOLUTE MINIMISERS

Herein we consider the problem of existence of minimisers and of 2nd order Absolute Minimisers for (1.1) with given boundary values (Definition 1). To this end we will assume that  $H$  is *level-convex* (namely has convex sub-level sets) and we will obtain our  $L^\infty$  objects in the limit of *approximate* minimisers of  $L^p$  functionals. The methods of this section have been inspired by the paper [BJW1] wherein the authors prove the existence of absolute minimisers when the rank of the gradient is at most one (scalar-valued functions or curves, see also the papers [K8, AK] for relevant ideas). We begin below with the simpler case of the existence of (mere) minimisers and subsequently we will show that the candidate we construct is indeed in  $AM^2(E_\infty, \Omega)$  when  $n = 1$ .

**Theorem 5** (Existence of  $L^\infty$  Minimisers and their  $L^p$ -approximation, I). *Let  $n \in \mathbb{N}$ ,  $\Omega \subseteq \mathbb{R}^n$  bounded and open and  $\mathbf{H} \in C(\mathbb{R}_s^{n \otimes 2})$  a non-negative level-convex function (that is for any  $t \geq 0$ , the set of matrices  $\{\mathbf{H} \leq t\}$  is convex in  $\mathbb{R}_s^{n \otimes 2}$ ). Suppose also there exist  $C_1, C_2, r > 0$  such that*

$$\mathbf{H}(X) \geq C_1|X|^r - C_2, \quad X \in \mathbb{R}_s^{n \otimes 2}.$$

*Then, for any  $g \in W^{2,\infty}(\Omega)$  there exists a function  $u_\infty \in W_g^{2,\infty}(\Omega)$  such that:*

(a)  *$u_\infty$  is a minimiser of (1.1) on  $\Omega$ , i.e.  $E_\infty(u, \Omega) \leq E_\infty(\psi, \Omega)$ , for any  $\psi \in W_g^{2,\infty}(\Omega)$ .*

(b) *For any  $q \geq 1$ ,  $u_\infty$  is the weak  $W^{2,q}(\Omega)$ -limit of a sequence of approximate minimisers  $(u_p)_{p=1}^\infty$  of the integral functionals (3.1) placed in  $W_g^{2,\infty}(\Omega)$  along a subsequence  $p_j \rightarrow \infty$ . Namely, for any  $q \geq 1$  we have  $u_{p_j} \rightharpoonup u_\infty$  in  $W^{2,q}(\Omega)$  as  $j \rightarrow \infty$  and  $u_p$  satisfies*

$$E_p(u_p, \Omega)^p \leq 2^{-p^2} + \inf \left\{ E_p(\cdot, \Omega)^p : W_g^{2,\infty}(\Omega) \right\}.$$

(c) *For any measurable  $A \subseteq \Omega$ , we have the “diagonal lower semi-continuity”*

$$E_\infty(u_\infty, A) \leq \liminf_{j \rightarrow \infty} E_{p_j}(u_{p_j}, A).$$

The proof of Theorem 5 can be done mutatis mutandis to the proof of Theorem 6 that follows and hence we refrain from giving the details. Theorem 5 does not include the case of the  $\infty$ -Bilaplacian (1.5) when minimising  $(\Delta u)^2$  and more generally when  $\mathbf{H}(X) = \mathbf{H}(A : X)$  for some matrix  $A > 0$ . In this case the appropriate space to obtain existence of minimisers is not  $W^{2,\infty}(\Omega)$  but instead the larger space

$$(5.1) \quad \mathcal{W}^{2,\infty}(\Omega) := \left\{ u \in \bigcap_{1 < p < \infty} W^{2,p}(\Omega) \mid A : D^2 u \in L^\infty(\Omega) \right\}$$

(with  $\mathcal{W}_0^{2,\infty}(\Omega)$  being defined in the obvious way) because of the inability to estimate  $D^2 u$  in terms of  $A : D^2 u$  in the  $L^\infty$  norm (see [GM]).

**Theorem 6** (Existence of  $L^\infty$  Minimisers and their  $L^p$ -approximation, II). *Let  $n \in \mathbb{N}$ ,  $\Omega \subseteq \mathbb{R}^n$  a bounded open set and  $\mathbf{H} \in C(\mathbb{R})$  a non-negative level-convex function (i.e. for any  $t \geq 0$ , the sets  $\{\mathbf{H} \leq t\}$  are intervals). Suppose also there exist  $C_1, C_2, r > 0$  such that*

$$\mathbf{H}(t) \geq C_1|t|^r - C_2, \quad t \in \mathbb{R}.$$

*Let also  $A \in \mathbb{R}_s^{n \otimes 2}$  be a strictly positive matrix. Then, for any  $g \in \mathcal{W}^{2,\infty}(\Omega)$  (see (5.1)) there exists a function  $u_\infty \in \mathcal{W}_g^{2,\infty}(\Omega)$  such that:*

(a)  *$u_\infty$  is a minimiser of the functional (3.1) for  $\mathbf{H}(X) = \mathbf{H}(A : X)$  over the space  $\mathcal{W}_g^{2,\infty}(\Omega)$ , i.e.  $E_\infty(u, \Omega) \leq E_\infty(\psi, \Omega)$  for any  $\psi \in \mathcal{W}_g^{2,\infty}(\Omega)$ .*

(b) *For any  $q \geq 1$ ,  $u_\infty$  is the weak  $W^{2,q}(\Omega)$ -limit of a sequence of approximate minimisers  $(u_p)_{p=1}^\infty$  of the integral functionals (3.1) (with  $\mathbf{H}(X) = \mathbf{H}(A : X)$ ) placed in  $\mathcal{W}_g^{2,\infty}(\Omega)$  along a subsequence  $p_j \rightarrow \infty$ . That is, for any  $q \geq 1$   $u_{p_j} \rightharpoonup u_\infty$  in  $W^{2,q}(\Omega)$  as  $j \rightarrow \infty$ , whilst  $u_p$  satisfies*

$$E_p(u_p, \Omega)^p \leq 2^{-p^2} + \inf \left\{ E_p(\cdot, \Omega)^p : \mathcal{W}_g^{2,\infty}(\Omega) \right\}.$$

(c) *The same lower semi-continuity statement as in Theorem 5(c) holds true but for the functionals (1.1) and (3.1) with  $\mathbf{H}(X) = \mathbf{H}(A : X)$ .*

The idea of the proof of Theorem 6 follows similar lines to those of the 1st order results of [BJW1, Theorem 2.1], [K8, Lemma 4], [K9, Lemma 5.1]. As in [BJW1], the essential point is the use of Young measures (valued in the Euclidean space, in contrast to the sphere-valued Young measure we employ in the definition of  $\mathcal{D}$ -solutions) in order to circumvent the lack of quasi-convexity for the  $L^p$  approximating functionals for which the infimum may not be attained at a minimiser (hence the need for approximate minimisers at the  $L^p$  level).

**Proof of Theorem 6.** We begin by noting that our coercivity lower bound and Hölder inequality imply the estimate

$$(5.2) \quad \frac{C_1}{2} \left( \int_{\Omega} |A : D^2 v|^k \right)^{\frac{r}{k}} - C_2 \leq \left( \int_{\Omega} \mathbf{H}(A : D^2 v)^p \right)^{\frac{1}{p}} \leq E_{\infty}(v, \Omega)$$

for any  $v \in \mathcal{W}_g^{2,\infty}(\Omega)$  and  $k \leq rp$ . Fix  $p > 1 + (1/r)$  and consider a minimising sequence  $(u_{p,i})_{i=1}^{\infty} \subseteq \mathcal{W}_g^{2,\infty}(\Omega)$  of  $E_p(\cdot, \Omega)^p$ . Select  $i = i(p)$  large so that

$$(5.3) \quad E_p(u_p, \Omega)^p \leq 2^{-p^2} + \inf \left\{ E_p(\cdot, \Omega)^p : \mathcal{W}_g^{2,\infty}(\Omega) \right\}$$

where  $u_p := u_{p,i(p)}$ . In particular,  $E_p(u_p, \Omega) \leq E_{\infty}(g, \Omega) + 1$  and by (5.2) the functions  $(A : D^2 u_p)_1^{\infty}$  are bounded in  $L^k(\Omega)$  for any fixed  $k \in \mathbb{N}$ . Since  $A$  is constant and  $u_p - g \in W_0^{2,k}(\Omega)$ , by the Calderon-Zygmund  $L^k$ -estimates (see e.g. [GT]) and Poincaré inequality, we have

$$\|A : D^2 u_p - A : D^2 g\|_{L^k(\Omega)} \geq C(k, n, A) \|u_p - g\|_{W^{2,k}(\Omega)}.$$

Thus, the sequence  $(u_p)_{p=1}^{\infty}$  is weakly precompact in  $W^{2,k}(\Omega)$  for any fixed  $k \in \mathbb{N}$  and there exists  $u_{\infty} \in \bigcap_{1 \leq k < \infty} W^{2,k}(\Omega)$  such that  $u_p \rightharpoonup u_{\infty}$  as  $p \rightarrow \infty$  along perhaps a subsequence. Further, by setting  $v = u_p$  in (5.2), using the weak lower-semicontinuity of the  $L^k(\Omega)$ -norm and letting  $p \rightarrow \infty$  and  $k \rightarrow \infty$ , we obtain  $A : D^2 u_{\infty} \in L^{\infty}(\Omega)$  which allows to infer that  $u_{\infty} \in \mathcal{W}_g^{2,\infty}(\Omega)$ . Further, by (5.3) and Hölder inequality, for any  $\psi \in \mathcal{W}_g^{2,\infty}(\Omega)$  and any  $q \leq p$  we have

$$(5.4) \quad \left( \int_{\Omega} \mathbf{H}(A : D^2 u_p)^q \right)^{\frac{1}{q}} \leq 2^{-p} + E_{\infty}(\psi, \Omega).$$

Consider now the sequence of Young measures generated by the scalar functions  $(A : D^2 u_p)_1^{\infty}$ , that is we consider  $(\delta_{A:D^2 u_p})_{p=1}^{\infty} \subseteq \mathcal{Y}(\Omega, \mathbb{R})$ . Then, along perhaps a further subsequence we have  $\delta_{A:D^2 u_p} \xrightarrow{*} \vartheta_{\infty}$  as  $p \rightarrow \infty$ . Since  $A : D^2 u_{\infty} \in L^{\infty}(\Omega)$ , the Young measure  $\vartheta_{\infty}$  has compact support in  $\mathbb{R}$ . Thus, there is  $R > 0$  such that  $\text{supp}(\vartheta_{\infty}(x)) \subseteq [-R, R]$  for a.e.  $x \in \Omega$ . Further, its barycentre is  $A : D^2 u_{\infty}$ , i.e.

$$A : D^2 u_{\infty}(x) = \int_{\mathbb{R}} t d[\vartheta_{\infty}(x)](t), \quad \text{a.e. } x \in \Omega.$$

Since  $\mathbf{H}$  is level-convex, by Jensen's inequality we have

$$\mathbf{H} \left( \int_{\mathbb{R}} t d[\vartheta_{\infty}(x)](t) \right) \leq [\vartheta_{\infty}(x)] - \text{ess sup}_{t \in \mathbb{R}} \mathbf{H}(t)$$

and since  $H$  is continuous and bounded from below

$$\|\mathbf{H}(A : D^2 u_{\infty})\|_{L^{\infty}(\Omega)} \leq \text{ess sup}_{x \in \Omega} \left\{ [\vartheta_{\infty}(x)] - \text{ess sup}_{t \in \mathbb{R}} \mathbf{H}(t) \right\}$$

$$\begin{aligned}
(5.5) \quad &= \lim_{q \rightarrow \infty} \left( \int_{\Omega} \int_{\mathbb{R}} \mathbf{H}(t)^q d[\vartheta_{\infty}(x)](t) dx \right)^{\frac{1}{q}} \\
&\leq \liminf_{q \rightarrow \infty} \left[ \liminf_{p \rightarrow \infty} \left( \int_{\Omega} \int_{\mathbb{R}} \mathbf{H}(t)^q d[\delta_{A:D^2 u_p}(x)](t) dx \right)^{\frac{1}{q}} \right] \\
&= \liminf_{q \rightarrow \infty} \left[ \liminf_{p \rightarrow \infty} \left( \int_{\Omega} \mathbf{H}(A : D^2 u_p)^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By combining (5.4)-(5.5), the conclusion of (a)-(b) follows. The proof of (c) is identical to that of [K8, Lemma 4].  $\square$

Now we establish a complete characterisation of 2nd order Absolute Minimisers and of the corresponding  $\mathcal{D}$ -solutions to the respective equation in the special case of  $n = 1$ .

**Theorem 7** (The fine structure of 2nd order Absolute Minimisers and of  $\mathcal{D}$ -solutions in 1D). *Let  $H \in C(\mathbb{R})$  with  $H \geq H(0) = 0$  and suppose that  $H$  is strictly level convex, that is  $H$  is strictly decreasing on  $(-\infty, 0)$  and strictly increasing on  $(0, \infty)$ . Further, suppose that there exist  $C_1, C_2 > 0$  and  $r > 1$  such that*

$$(5.6) \quad H(X) \geq C_1 |X|^r - C_2, \quad X \in \mathbb{R}.$$

*Let  $T^- < 0 < T^+$  denote the elements of the level set  $\{H = t\}$  for  $t > 0$ :  $\{T^-, T^+\} = \{X \in \mathbb{R} : H(X) = t\}$ . We also suppose that*

$$(5.7) \quad 0 < \liminf_{t \rightarrow \infty} \left| \frac{T^-}{T^+} \right| \leq \limsup_{t \rightarrow \infty} \left| \frac{T^-}{T^+} \right| < \infty.$$

*Let also  $a < b$  in  $\mathbb{R}$ . We consider the functional (1.1) on  $(a, b)$ , that is*

$$E_{\infty}(u, \mathcal{O}) = \|H(u'')\|_{L^{\infty}(\mathcal{O})}, \quad u \in W^{2,\infty}(a, b), \quad \mathcal{O} \subseteq (a, b).$$

*If  $H \in C^1(\mathbb{R})$  we consider also the corresponding equation (1.2), that is*

$$A_{\infty}^2 u = (H_X(u''))^3 (u''')^2 = 0, \quad \text{on } (a, b).$$

*Let also  $g \in W^{2,\infty}(a, b)$  and set*

$$(5.8) \quad \mathcal{E}(g) := \frac{g'(b) - g'(a)}{b - a} - \frac{2(g(b) - g(a) - g'(a)(b - a))}{(b - a)^2}.$$

*Then, we have:*

(1) *There exists a unique 2nd order Absolute Minimiser  $u_* \in \text{AM}^2(E_{\infty}, (a, b)) \cap W_g^{2,\infty}(a, b)$  with given boundary values. Moreover, if  $\mathcal{E}(g) = 0$  then  $u_*$  is a quadratic polynomial function. If  $\mathcal{E}(g) \neq 0$ , then  $u_*$  is piecewise quadratic with exactly one point  $\xi_* \in (a, b)$  at which  $u_*''$  does not exist and with  $u_*''$  changing sign at  $\xi_*$ . Further,  $H(u_*'')$  extends to a constant function on  $(a, b)$ . Moreover,  $u_*$  coincides with the limit function  $u_{\infty}$  as  $p \rightarrow \infty$  of approximate  $L^p$  minimisers of Theorem 5.*

(2) *Every 2nd order Absolute Minimiser  $u \in \text{AM}^2(E_{\infty}, (a, b))$  has the structure described by (1) above, i.e.  $u$  is quadratic if  $\mathcal{E}(u) = 0$  and is piecewise quadratic with one point at which  $u''$  does not exist and changes sign if  $\mathcal{E}(u) \neq 0$ . Also,  $H(u'')$  extends to a constant function on  $(a, b)$  and  $u$  coincides with the limit of approximate  $L^p$  minimisers in  $W_u^{2,\infty}(a, b)$  as  $p \rightarrow \infty$ .*

(3) If in addition  $H \in C^1(\mathbb{R})$ , the Dirichlet problem

$$(5.9) \quad \begin{cases} A_\infty^2 u = 0, & \text{in } (a, b), \\ u = g, \quad u' = g', & \text{at } \{a, b\}, \end{cases}$$

has a unique Absolutely Minimising  $\mathcal{D}$ -solution  $u_\infty \in W_g^{2,\infty}(a, b)$  which is piecewise quadratic with at most one point in  $(a, b)$  at which  $u_\infty''$  may not exist. Further, every Absolutely Minimising  $\mathcal{D}$ -solution to the problem (5.9) is unique has this form.

(4) Every 2nd order Absolute Minimiser of (1.1) for  $n = 1$  is a  $\mathcal{D}$ -solution to (1.2).

**Remark 8.** i) The assumption (5.7) requires that “the growth of  $H$  at  $+\infty$  can not be too far away from the growth of  $H$  at  $-\infty$ ”. It is satisfied for instance if  $H(-X) = H(\alpha X + o(|X|))$  for some  $\alpha > 0$  as  $|X| \rightarrow \infty$ .

ii) The condition  $\mathcal{E}(g) = 0$  (where  $\mathcal{E}(g)$  is given by (5.8)) is necessary and sufficient for the existence of a quadratic polynomial  $Q$  with  $Q - g \in W_0^{2,\infty}(a, b)$ .

iii) The converse of item (4) is not true in general (see Section 8).

The proof of Theorem 7 consists of several lemmas. We begin by recording the following simple observation which relates to Aronsson’s result [A1, Lemma 1, p. 34] and is an immediate consequence of the mean value theorem:

**Remark 9.** Suppose that  $u$  is a quadratic polynomial on  $(\alpha, \beta) \subseteq \mathbb{R}$  with  $u'' \equiv C$  and  $\phi \in W^{2,\infty}(\alpha, \beta)$  with  $\phi \not\equiv u$  and  $\phi' = u'$  at  $\{\alpha, \beta\}$ . Then, there exist measurable sets  $A^\pm \subseteq (\alpha, \beta)$  with  $\mathcal{L}^1(A^\pm) > 0$  (positive Lebesgue measure) such that  $\phi''$  exists on  $A^+ \cup A^-$ , whilst we have  $\phi'' > C$  on  $A^+$  and  $\phi'' < C$  on  $A^-$ .

We first consider the much simpler case of  $\mathcal{E}(g) = 0$ .

**Lemma 10.** Every quadratic polynomial  $u : \mathbb{R} \rightarrow \mathbb{R}$  is the unique minimiser of  $E_\infty$  over  $W_u^{2,\infty}(a, b)$  with respect to its own boundary conditions.

**Proof of Lemma 10.** Suppose  $\phi \in W_u^{2,\infty}(a, b) \setminus \{u\}$  and  $u'' \equiv C$  on  $(a, b)$ . If  $C \geq 0$ , by Remark 9 there is a measurable set  $A^+ \subseteq (a, b)$  with  $\mathcal{L}^1(A^+) > 0$  such that  $\phi'' > C \geq 0$  on  $A^+$ . Since  $H$  is strictly increasing on  $(0, \infty)$ , we have

$$E_\infty(\phi, (a, b)) \geq E_\infty(\phi, A^+) = \|H(\phi'')\|_{L^\infty(A^+)} > H(C) = E_\infty(u, (a, b)).$$

The case  $C < 0$  follows analogously since by Remark 9 there is a measurable  $A^- \subseteq (a, b)$  with  $\mathcal{L}^1(A^-) > 0$  and  $\phi'' < C < 0$ , whilst  $H$  is strictly decreasing on  $(-\infty, 0)$  and we again obtain  $E_\infty(\phi, (a, b)) > E_\infty(u, (a, b))$ . Uniqueness follows by the strictness of the energy inequalities.  $\square$

We now consider the case of  $\mathcal{E}(g) \neq 0$ .

**Lemma 11.** If  $\mathcal{E}(g) \neq 0$ , there exists a unique piecewise quadratic function  $u_* \in W_g^{2,\infty}(a, b)$  and a  $\xi_* \in (a, b)$  such that  $u'' \equiv L_*$  on  $(a, \xi_*)$ ,  $u'' \equiv R_*$  on  $(\xi_*, b)$  and

$$\|H(u_*'')\|_{L^\infty(a, b)} = \max\{H(L_*), H(R_*)\}, \quad H(L_*) = H(R_*), \quad L_* R_* < 0.$$

**Proof of Lemma 11.** For brevity we set  $g(a) = A$ ,  $g(b) = B$ ,  $g'(a) = A'$  and  $g'(b) = B'$ . We first show that a piecewise quadratic function in  $W_g^{2,\infty}(a, b)$  with one matching point indeed exists. We fix parameters  $R, L \in \mathbb{R}$  and set

$$\begin{cases} u^L(x) := A + A'(x - a) + \frac{L}{2}(x - a)^2, \\ u^R(x) := B + B'(x - b) + \frac{R}{2}(x - b)^2. \end{cases}$$

The condition of  $C^1$  matching of  $u^L, u^R$  at some  $\xi \in (a, b)$  to a single function  $u = u^L \chi_{(a, \xi)} + u^R \chi_{[\xi, b]}$  in  $W_g^{2, \infty}(a, b)$  is equivalent to

$$(5.10) \quad A + A'(\xi - a) + \frac{L}{2}(\xi - a)^2 = B + B'(\xi - b) + \frac{R}{2}(\xi - b)^2,$$

$$(5.11) \quad (L - R)\xi = B' - A' + aL - bR.$$

Since by assumption  $\mathcal{E}(g) \neq 0$ , it follows that  $R \neq L$ . We now set

$$(5.12) \quad C_0 := \frac{B' - A'}{b - a}, \quad C_1 := \frac{2(A - B - B'(a - b))}{(b - a)^2}, \quad C_2 := \frac{2(B - A - A'(b - a))}{(b - a)^2}.$$

By (5.11)-(5.12), the constraint  $a < \xi < b$  is equivalent to

$$(5.13) \quad \begin{cases} R < C_0 < L, & \text{if } R < L, \\ L < C_0 < R, & \text{if } L < R. \end{cases}$$

By cancelling  $\xi$  from (5.10) with the aid of (5.11), we obtain that the admissible pairs  $(R, L) \in \mathbb{R}^2$  for matching lie on the hyperbola  $\mathcal{C} \subseteq \mathbb{R}^2$  given by the equation

$$(5.14) \quad (R - C_1)(L - C_2) = C_1 C_2 - C_0^2.$$

Then, (5.14) coincides with the condition of vanishing discriminant of the algebraic equation (5.10) (when considered as a binomial equation with respect to  $\xi$ ) and the unique point  $\xi$  for matching is that given by (5.11). Note that the lines  $\{R = C_1\}$  and  $\{L = C_2\}$  are the asymptotes of  $\mathcal{C}$ . Since

$$\mathcal{E}(g) = \frac{B' - A'}{b - a} - \frac{2(B - A - A'(b - a))}{(b - a)^2} = C_0 - C_2$$

the following facts can be easily verified by elementary algebraic calculations:

$$(5.15) \quad \mathcal{E}(g) > 0 \iff C_1 > C_0 > C_2,$$

$$(5.16) \quad C_0^2 - C_1 C_2 = \mathcal{E}(g)^2 > 0,$$

$$(5.17) \quad (C_0, C_0) \in \mathcal{C} \text{ \& \{ } R = L \text{ \} is tangent to } \mathcal{C} \text{ at } (C_0, C_0).$$

Note further that  $\mathcal{C}$  lies on the 2nd and 4th quadrants of  $\mathbb{R}^2$ , i.e.  $(R - C_1)(L - C_2) < 0$  for all  $(R, L) \in \mathcal{C}$ . We now derive the remaining constraints that  $(R, L)$  have to satisfy in order to be admissible. We set  $\delta := b - \xi$  and rewrite (5.10)-(5.11) as

$$\begin{cases} \delta^2(L - R) - 2\delta[L(b - a) - (B' - A')] + L(b - a)^2 = 2(B - A - A'(b - a)), \\ \delta(L - R) = L(b - a) - (B' - A'). \end{cases}$$

The above imply the inequality

$$(5.18) \quad \delta^2 = \frac{(b - a)^2}{L - R} \left( L - \frac{2(B - A - A'(b - a))}{(b - a)^2} \right) \geq 0.$$

Similarly, we set  $\varepsilon := \xi - a$  and rewrite (5.10)-(5.11) as

$$\begin{cases} \varepsilon^2(L - R) + 2\varepsilon[R(b - a) - (B' - A')] - R(b - a)^2 = -2(A - B - B'(a - b)), \\ -\varepsilon(L - R) = R(b - a) - (B' - A'). \end{cases}$$

The above imply the inequality

$$(5.19) \quad \varepsilon^2 = \frac{(b-a)^2}{R-L} \left( R - \frac{2(A-B-B'(a-b))}{(b-a)^2} \right) \geq 0.$$

By (5.19), (5.18), (5.13) and (5.12), we have that the admissible pairs  $(R, L)$  lie on the constraint set

$$(5.20) \quad \mathcal{K} := \left\{ (R, L) \in \mathbb{R}^2 \mid \begin{array}{l} R \leq C_1, \ L \geq C_2 \text{ and } R < C_0 < L, \text{ if } R < L; \\ R \geq C_1, \ L \leq C_2 \text{ and } L < C_0 < R, \text{ if } L < R \end{array} \right\}.$$

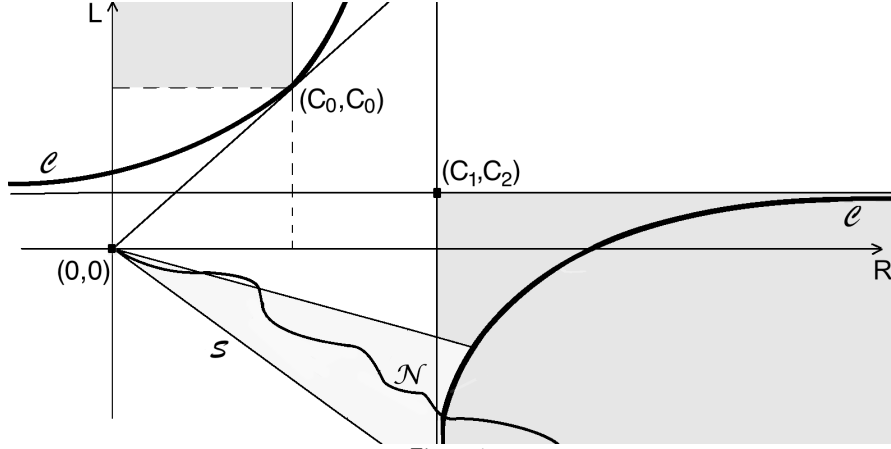


Figure 1.

We consider first the case  $\mathcal{E}(g) > 0$ . By (5.15), the set of admissible pairs is

$$\mathcal{C} \cap \mathcal{K} = \left\{ (R, L) \in \mathcal{C} : R \geq C_1, \ L \leq C_2 \right\}.$$

Consider now the continuous curve

$$\mathcal{N} := \left\{ (R, L) \in \mathbb{R}^2 : L < 0 < R, \ H(L) = H(R) \right\}.$$

By our assumptions on  $H$ , the correspondences  $R \mapsto L$  and  $L \mapsto R$  are inverse of each other and hence  $\mathcal{N}$  is the graph of a monotone function. By our assumption (5.7), for  $|(R, L)|$  large enough  $\mathcal{N}$  lies in a sector of the form

$$\mathcal{S} := \left\{ (R, L) \in \mathbb{R}^2 : \left| \angle((R, L), (1, -1)) \right| \leq a \right\}$$

for some  $a < \pi/4$ . Since  $\mathcal{C} \cap \mathcal{K}$  intersects every ray in the sector  $\mathcal{S}$  emanating from the origin, it follows that there is a unique point of intersection  $(R_*, L_*) \in \mathcal{N} \cap \mathcal{C} \cap \mathcal{K}$  giving rise to a unique matching point  $\xi_*$  given by (5.11). Then, the function  $u_* := u^{L_*} \chi_{(a, \xi_*)} + u^{R_*} \chi_{[\xi_*, b)}$  has the desired properties. The case of  $\mathcal{E}(g) < 0$  follows analogously, so the proof of the lemma is complete.  $\square$

**Lemma 12.** *If  $\mathcal{E}(g) \neq 0$ , the function  $u_* \in W_g^{2,\infty}(a, b)$  obtained in Lemma 11 is the unique minimiser of  $E_\infty$  over  $W_g^{2,\infty}(a, b)$ .*

**Proof of Lemma 12.** We simplify the notation and symbolise  $u_*$  by just  $u$ . Fix  $v \in W_g^{2,\infty}(a, b) \setminus \{u\}$ . Then,  $v' \in W_{u'}^{1,\infty}(a, b)$  and  $v' \not\equiv u'$ . Without loss of generality consider the case  $R < 0 < L$ . If  $v'$  intersects  $u'$  at some point in  $(a, \xi)$ , then by



Remark 9 there exists  $A^+ \subseteq (a, \xi)$  with  $\mathcal{L}^1(A^+) > 0$  such that  $v'' > L > 0$  on  $A^+$ . Since  $H$  is strictly increasing on  $(0, \infty)$ , we have

$$\begin{aligned} E_\infty(v, (a, b)) &\geq E_\infty(v, A^+) = \operatorname{ess\,sup}_{A^+} H(v'') > \\ &> H(L) = \max\{H(L), H(R)\} = E_\infty(u, (a, b)). \end{aligned}$$

Similarly, if  $v'$  intersects  $u'$  at some point in  $[\xi, b)$ , then by Remark 9 there exists  $A^- \subseteq (\xi, b)$  with  $\mathcal{L}^1(A^-) > 0$  such that  $v'' < R < 0$  on  $A^-$ . Since  $H$  is strictly decreasing on  $(-\infty, 0)$ , we again obtain  $E_\infty(v, (a, b)) > E_\infty(u, (a, b))$ . Hence, it remains to consider the cases that  $v'$  lies either above or below  $u'$  over  $(a, b)$ . The former case can also be handled by Remark 9: Since  $v'(a) = u'(a)$  and  $v'(\xi) > u'(\xi)$ , there is a set  $A^+ \subseteq (a, \xi)$  with  $\mathcal{L}^1(A^+) > 0$  such that

$$v'' > \frac{v'(\xi) - v'(a)}{\xi - a} > \frac{u'(\xi) - u'(a)}{\xi - a} = L > 0, \quad \text{on } A^+,$$

and again  $E_\infty(v, (a, b)) > E_\infty(u, (a, b))$ . Finally, if  $v' < u'$  on  $(a, b)$ , then since  $u(a) = v(a)$  by integration we get  $v < u$  on  $(a, b)$ . Since  $u$  is quadratic on  $(\xi, b)$  with  $u'' = R$  and  $v(\xi) < u(\xi)$ , by Taylor's theorem we have

$$\begin{aligned} v(b) + v'(b)(\xi - b) + \frac{1}{2} \left( \int_0^1 v''(b + t(\xi - b)) [2(1 - t)] dt \right) (\xi - b)^2 \\ < u(b) + u'(b)(\xi - b) + \frac{R}{2} (\xi - b)^2. \end{aligned}$$

Since  $u(b) = v(b) = B$ ,  $u'(b) = v'(b) = B'$ , by considering the absolutely continuous probability measure  $\mu \ll \mathcal{L}^1$  on  $[0, 1]$  given by  $\mu(E) := \int_E 2(1 - t) dt$ , we deduce

$$\int_0^1 v''(b + t(\xi - b)) d\mu(t) < R \mu([0, 1]).$$

Hence, there exists a measurable set  $A \subseteq (0, 1)$  with  $\mathcal{L}^1(A) > 0$  such that  $v''(b + t(\xi - b)) < R$  for all points  $t \in A$ . Since  $R < 0$ , by arguing as before we obtain  $E_\infty(v, (a, b)) > E_\infty(u, (a, b))$ . The lemma ensues.  $\square$

**Lemma 13.** *In the setting of Theorem 5, the function  $u_\infty$  constructed therein is a 2nd order absolute minimiser of  $E_\infty$  in  $W_g^{2,\infty}(a, b)$ .*

**Proof of Lemma 13.** We begin with the following observation: given  $A, B, A', B'$ ,  $a, b \in \mathbb{R}$  with  $a < b$ , the unique cubic Hermite interpolant  $Q : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $Q(a) = A$ ,  $Q(b) = B$ ,  $Q'(a) = A'$  and  $Q'(b) = B'$ . Further, let  $(v_p)_1^\infty \subseteq W^{2,\infty}(a, b)$  be any sequence of functions satisfying  $v_p \rightarrow v_\infty$  in  $C^1[a, b]$  as  $p \rightarrow \infty$ . If  $Q_p$  is the cubic polynomial such that  $Q_p - v_p \in W_0^{2,\infty}(a, b)$  for  $p \in \mathbb{N} \cup \{\infty\}$ , namely

$$Q_p(a) = v_p(a), \quad Q_p(b) = v_p(b), \quad Q'_p(a) = v'_p(a), \quad Q'_p(b) = v'_p(b),$$

we have  $Q_p \rightarrow Q_\infty$  strongly in  $W^{2,\infty}(a, b)$  as  $p \rightarrow \infty$ .

We now continue with the existence of an absolute minimiser. Let  $(u_p)_{p=1}^\infty$  be the sequence of approximate minimisers of Theorem 5 which satisfies  $u_p \rightarrow u_\infty$  in  $W^{2,q}(\Omega)$  as  $p \rightarrow \infty$  along a subsequence for any  $q > 1$ . Fix an  $\Omega' \Subset \Omega$  and  $\phi \in W_0^{2,\infty}(\Omega')$ . Since any open set on  $\mathbb{R}$  is a countable disjoint union of intervals, we may assume  $\Omega' = (a, b)$ . In order to conclude, it suffices to show that

$$(5.21) \quad E_\infty(u_\infty, (a, b)) \leq E_\infty(u_\infty + \phi, (a, b)),$$

when  $\phi \in W_0^{2,\infty}(a, b)$ . Consider for any  $p \in \mathbb{N} \cup \{\infty\}$  the unique cubic polynomial such that  $Q_p - u_p \in W_0^{2,\infty}(a, b)$ . By the above observations and Theorem 5(a)-(b), along a subsequence  $p_j \rightarrow \infty$  we have

$$(5.22) \quad Q_p \longrightarrow Q_\infty \quad \text{in } W^{2,\infty}(a, b), \quad \text{as } p \rightarrow \infty.$$

We define for  $p \in \mathbb{N}$  the function  $\phi_p := \phi + u_\infty - u_p + Q_p - Q_\infty$ . Since all three functions  $\phi$ ,  $u_\infty - Q_\infty$  and  $u_p - Q_p$  are in  $W_0^{2,\infty}(a, b)$ , the same is true for  $\phi_p$  as well. By Theorem 5(b) and by the additivity of the integral, we have

$$(5.23) \quad \begin{aligned} E_p(u_p, (a, b)) &\leq 2^{-p} + E_p(u_p + \phi_p, (a, b)) \\ &= 2^{-p} + E_p(u_\infty + \phi + [Q_p - Q_\infty], (a, b)) \\ &\leq 2^{-p} + \left( \frac{b-a}{\mathcal{L}^1(\Omega)} \right)^{1/p} E_\infty(u_\infty + \phi + [Q_p - Q_\infty], (a, b)). \end{aligned}$$

By invoking (5.22) and passing to the limit in (5.23) as  $p \rightarrow \infty$ , we deduce (5.21) as a consequence of Theorem 5(c). The lemma ensues.  $\square$

We can now prove the result by using the above lemmas.

**Proof of Theorem 7.** (1) By Lemmas 10-12, there exists a unique minimiser  $u_*$  of  $E_\infty$  in  $W_g^{2,\infty}(a, b)$  which is either quadratic or piecewise quadratic with at most one breaking point for  $u_*$  at which it changes sign and  $H(u_*)$  extends to a constant function on  $(a, b)$ . By Lemma 13, the limit  $u_\infty$  as  $p \rightarrow \infty$  of approximate  $L^p$  minimisers is a 2nd order Absolute Minimiser and a fortiori a minimiser of  $E_\infty$  in  $W_g^{2,\infty}(a, b)$ . Thus,  $u_* \equiv u_\infty$  and this is the unique element of  $\text{AM}^2(E_\infty, (a, b)) \cap W_g^{2,\infty}(a, b)$ . (2) is a consequence of (1). (3) and (4) follow by the fact that  $H(u_*) = C$  a.e. on  $(a, b)$ , uniqueness and Claim 20 of Section 7 that follows. The theorem ensues.  $\square$

## 6. VARIATIONAL CHARACTERISATION OF $A_\infty^2$ VIA $\text{AM}^2$ & UNIQUENESS

Herein we show that 2nd order Absolute Minimisers of (1.1) in  $C^3(\Omega)$  solve the fully nonlinear PDE (1.2). The converse is also true in the case that  $H$  depends on the hessian via a scalar projection of it along a matrix. Let us note that in that case by the Spectral Theorem and a change of variables, the study of this functional can be reduced to the study of one depending on the hessian via *just* the Laplacian, but we find it more elucidating to retain this seemingly more general form. As a consequence, we deduce the uniqueness of  $C^3$  2nd order Absolute Minimisers and of classical solutions to the PDE. Accordingly, the main result of this section is:

**Theorem 14** (Variational characterisation of  $A_\infty^2$  via  $E_\infty$ ). *Given a non-negative function  $H \in C^1(\mathbb{R}_s^{n \otimes 2})$ , an open set  $\Omega \subseteq \mathbb{R}^n$  and  $u \in C^3(\Omega)$ , consider the supremal functional  $E_\infty$  given by (1.1) and the fully nonlinear equation  $A_\infty^2 u = 0$  given by (1.2). Then:*

(A) *If  $u \in \text{AM}^2(E_\infty, \Omega) \cap C^3(\Omega)$ , namely if it is a  $C^3$  2nd order Absolute Minimiser (Definition 1), then it solves  $A_\infty^2 u = 0$  on  $\Omega$ .*

(B) *Suppose that  $\Omega$  is connected and  $H$  has the form  $H(X) = \mathbf{H}(A:X)$  for a fixed (strictly) positive matrix  $A \in \mathbb{R}_s^{n \otimes 2}$  and some level-convex function  $\mathbf{H} \in C^1(\mathbb{R})$  (that is  $\mathbf{H}$  has for any  $t \geq 0$  convex sub-level sets  $\{\mathbf{H} \leq t\}$ ) such that  $\{\mathbf{H} = t\}$  consists of at most 2 points. Then, the statements (1)-(4) below are equivalent:*

- (1)  $u \in \text{AM}^2(\text{E}_\infty, \Omega)$ .
- (2) There exists  $C \geq 0$  such that  $A : \text{D}^2 u \equiv C$  in  $\Omega$ .
- (3) There exists  $c \geq 0$  such that  $\mathbf{H}(A : \text{D}^2 u) \equiv c$  in  $\Omega$ .
- (4)  $\text{A}_\infty^2 u = 0$  in  $\Omega$ .

As a consequence, we deduce the next result:

**Corollary 15** (Uniqueness of  $C^3$  Absolute Minimisers and of  $C^3$  solutions to the Dirichlet problem). *In the setting of Theorem 14(B) above, suppose in addition  $\Omega$  is bounded. Then, for any  $g \in W^{2,\infty}(\Omega)$ , the intersection*

$$C^3(\Omega) \cap W_g^{2,\infty}(\Omega) \cap \text{AM}^2(\text{E}_\infty, \Omega)$$

*contains at most one element, namely there is at most one  $C^3$  Absolute Minimiser  $u$  of 2nd order which satisfies  $u = g$  and  $\text{Du} = \text{D}g$  on  $\partial\Omega$ . Further, the problem*

$$(6.1) \quad \begin{cases} \left( \mathbf{H}'(A : \text{D}^2 u) A \right)^{\otimes 3} : (\text{D}^3 u)^{\otimes 2} = 0, & \text{in } \Omega, \\ u = g, \text{ Du} = \text{D}g, & \text{on } \partial\Omega, \end{cases}$$

*has at most one solution in  $C^3(\Omega) \cap W_g^{2,\infty}(\Omega)$ .*

We begin with a simple lemma which is relevant to some of the results of [K10].

**Lemma 16.** *Let  $\mathbf{H} \in C^1(\mathbb{R}_s^{n \otimes 2})$  be given and consider the functional (1.1). Let also  $\Omega \subseteq \mathbb{R}^n$  be an open set.*

(a) *For any  $u \in C^2(\Omega)$  and  $\mathcal{O} \Subset \Omega$ , we set*

$$(6.2) \quad \mathcal{O}(u) := \left\{ x \in \overline{\mathcal{O}} : \mathbf{H}(\text{D}^2 u(x)) = \|\mathbf{H}(\text{D}^2 u)\|_{L^\infty(\mathcal{O})} \right\}.$$

*Then, if  $u \in \text{AM}^2(\text{E}_\infty, \Omega)$  (Definition 1), it follows that*

$$\mathbf{H}_X(\text{D}^2 u) : \text{D}^2 \phi = 0, \quad \text{on } \mathcal{O}(u),$$

*for any  $\phi \in W_0^{2,\infty}(\mathcal{O}) \cap C^2(\overline{\mathcal{O}})$ .*

(b) *For any  $u \in C^2(\Omega)$ ,  $x \in \Omega$  and  $0 < \varepsilon < \frac{1}{2} \text{dist}(x, \partial\Omega)$ , we set*

$$(6.3) \quad \Omega_\varepsilon(x) := \mathbb{B}_\varepsilon(x) \cap \left\{ \mathbf{H}(\text{D}^2 u) < \mathbf{H}(\text{D}^2 u(x)) \right\}$$

*and suppose  $\Omega_\varepsilon(x) \neq \emptyset$ . Then,  $x \in \partial(\Omega_\varepsilon(x))$  and*

$$\mathbf{H}(\text{D}^2 u(y)) = \|\mathbf{H}(\text{D}^2 u)\|_{L^\infty(\Omega_\varepsilon(x))}$$

*for all  $y \in \mathbb{B}_\varepsilon(x) \cap \partial(\Omega_\varepsilon(x))$  (in particular for  $y = x$ ).*

**Proof of Lemma 16.** (a) This is an application of Danskin's theorem [Da]. Fix  $\mathcal{O} \Subset \Omega$  and  $\phi \in W_0^{2,\infty}(\mathcal{O}) \cap C^2(\overline{\mathcal{O}})$ . If  $u \in \text{AM}^2(\text{E}_\infty, \Omega)$ , then the real function  $t \mapsto \|\mathbf{H}(\text{D}^2 u + t\text{D}^2 \phi)\|_{L^\infty(\mathcal{O})}$  attains a local minimum at  $t = 0$ , so if its derivative exists it must vanish. By Danskin's theorem we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \left( \|\mathbf{H}(\text{D}^2 u + t\text{D}^2 \phi)\|_{L^\infty(\mathcal{O})} \right) &= \frac{d}{dt} \Big|_{t=0} \left( \max_{\overline{\mathcal{O}}} \mathbf{H}(\text{D}^2 u + t\text{D}^2 \phi) \right) = \\ &= \max_{x \in \mathcal{O}(u)} \left( \frac{d}{dt} \Big|_{t=0} \mathbf{H}(\text{D}^2 u + t\text{D}^2 \phi)(x) \right) = \max_{x \in \mathcal{O}(u)} \left( \mathbf{H}_X(\text{D}^2 u(x)) : \text{D}^2 \phi(x) \right) \end{aligned}$$

and upon replacing  $\phi$  with  $-\phi$ , the conclusion follows.

(b) This is obvious from the definitions.  $\square$

**Proof of Theorem 14.** (A) Fix  $x \in \Omega$  and  $\varepsilon \in (0, \text{dist}(x, \partial\Omega))$ . If  $\Omega_\varepsilon(x) = \emptyset$ , then we have  $H(D^2u) \geq H(D^2u(x))$  on  $\mathbb{B}_\varepsilon(x)$ , which implies that  $x$  is an interior minimum, therefore giving  $D(H(D^2u))(x) = 0$ . Hence,  $A_\infty^2 u(x) = 0$ . Consequently, we may assume  $\Omega_\varepsilon(x) \neq \emptyset$ . By (a) and (b) of Lemma 16 above, if  $u \in \text{AM}^2(E_\infty, \Omega) \cap C^2(\Omega)$ , we have

$$(6.4) \quad H_X(D^2u(x)) : D^2\phi(x) = 0$$

for any  $\phi \in W_0^{2,\infty}(\Omega_\varepsilon(x)) \cap C^2(\overline{\Omega_\varepsilon(x)})$ . We illustrate the idea of the rest of the proof by showing it first under one more degree of regularity.

**Claim 17.** *Theorem 14(A) is true if in addition  $u \in C^4(\Omega)$  and  $H \in C^2(\mathbb{R}_s^{n \otimes 3})$ .*

**Proof of Claim 17.** Fix  $x, \varepsilon, \phi$  as above. If  $\Omega_\varepsilon(x) \neq \emptyset$ , let  $\zeta \in C_c^\infty(\mathbb{B}_\varepsilon(x))$  be a cut off function with  $\zeta \geq 0$  on  $\mathbb{B}_\varepsilon(x)$  and  $\zeta \equiv 1$  on  $\mathbb{B}_{\varepsilon/2}(x)$ . We define

$$(6.5) \quad \phi := \frac{1}{2} \left( H(D^2u) - H(D^2u(x)) \right)^2 \zeta.$$

Then,  $\phi \in C^2(\overline{\Omega_\varepsilon(x)})$ ,  $\phi = 0$  on  $\partial(\Omega_\varepsilon(x))$  and on  $\overline{\Omega_\varepsilon(x)}$  we have

$$(6.6) \quad D\phi = \zeta \left( H(D^2u) - H(D^2u(x)) \right) D(H(D^2u)) + \left( H(D^2u) - H(D^2u(x)) \right)^2 D\zeta$$

whilst on  $\mathbb{B}_{\varepsilon/2}(x)$  we have

$$D^2\phi = D(H(D^2u)) \otimes D(H(D^2u)) + \left( H(D^2u) - H(D^2u(x)) \right) D^2(H(D^2u)).$$

Hence,  $D\phi = 0$  on  $\partial(\Omega_\varepsilon(x))$  and

$$D^2\phi(x) = D(H(D^2u))(x) \otimes D(H(D^2u))(x).$$

By inserting the above to (6.4), we obtain

$$A_\infty^2 u(x) = H_X(D^2u(x)) : D(H(D^2u))(x) \otimes D(H(D^2u))(x) = 0,$$

for any  $x \in \Omega$ . The claim ensues.  $\square$

Now we complete the proof of (A) by considering the case of “natural” regularity  $u \in \text{AM}^2(E_\infty, \Omega) \cap C^3(\Omega)$  and  $H \in C^1(\mathbb{R}_s^{n \otimes 3})$ . Let again  $\phi$  and  $\zeta$  be as in (6.5). Then, we have  $\phi \in C^1(\overline{\Omega_\varepsilon(x)})$ ,  $\phi = 0$  on  $\partial(\Omega_\varepsilon(x))$  and (6.6) still holds giving  $D\phi = 0$  on  $\partial(\Omega_\varepsilon(x))$ . The problem is that  $D^2(H(D^2u))$  may not exist.

However, we now show that  $H(D^2u)$  is twice differentiable in the sense of Whitney (see [W, M, F]) on the closed set  $\overline{\mathbb{B}_\varepsilon(x)} \cap \partial(\Omega_\varepsilon(x))$  and hence it admits a  $C^2$  extension which coincides with it up to 2nd order on  $\overline{\mathbb{B}_\varepsilon(x)} \cap \partial(\Omega_\varepsilon(x))$ . To this end we need to introduce some notation. Given any functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ , we set

$$D^{1,z}f(y) := \frac{f(y+z) - f(y)}{|z|}, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad z \neq 0,$$

and note the elementary identity

$$(6.7) \quad D^{1,z}(fg)(y) = f(y) D^{1,z}g(y) + g(y+z) D^{1,z}f(y).$$

Let us also set for brevity  $\mathcal{H} := H(D^2u) - H(D^2u(x))$  and  $\mathcal{K} := \overline{\mathbb{B}_\varepsilon(x)} \cap \partial(\Omega_\varepsilon(x))$ . By applying (6.7), we have

$$D^{1,z}D\left(\frac{1}{2}\mathcal{H}^2\right) = D^{1,z}(\mathcal{H} D\mathcal{H}) = \mathcal{H} (D^{1,z}D\mathcal{H}) + (D^{1,z}\mathcal{H}) D\mathcal{H}(\cdot + z).$$

Note now that (6.3) yields  $\mathcal{H} \equiv 0$  on  $\mathcal{K}$ . Hence, we obtain

$$(6.8) \quad D^{1,z}D\left(\frac{1}{2}\mathcal{H}^2\right) = (D^{1,z}\mathcal{H})D\mathcal{H}(\cdot+z), \quad \text{on } \mathcal{K}.$$

We now claim that there exists an increasing modulus of continuity  $\omega \in C[0, \infty)$  with  $\omega(0) = 0$  such that

$$(6.9) \quad \max_{y \in \mathcal{K}} \left| \frac{D(\frac{1}{2}\mathcal{H}^2)(z+y) - D(\frac{1}{2}\mathcal{H}^2)(y) - (D\mathcal{H}(y) \otimes D\mathcal{H}(y))z}{|z|} \right| \leq \omega(|z|)$$

for any  $0 < |z| < \varepsilon/2$ . In order to establish (6.9), we fix a  $y \in \mathcal{K}$  and calculate:

$$\begin{aligned} & \left| \frac{D(\mathcal{H}^2)(z+y) - D(\mathcal{H}^2)(y) - 2(D\mathcal{H}(y) \otimes D\mathcal{H}(y))z}{|z|} \right| \\ &= \left| D^{1,z}D(\mathcal{H}^2)(y) - 2 \frac{D\mathcal{H}(y) \otimes D\mathcal{H}(y)}{|z|} z \right| \\ &\stackrel{(6.8)}{=} 2 \left| D^{1,z}\mathcal{H}(y)D\mathcal{H}(y+z) - \left( D\mathcal{H}(y) \otimes D\mathcal{H}(y) \right) \frac{z}{|z|} \right| \\ &= 2 \left| \frac{\mathcal{H}(y+z) - \mathcal{H}(y)}{|z|} D\mathcal{H}(y+z) - \left( D\mathcal{H}(y) \cdot \frac{z}{|z|} \right) D\mathcal{H}(y) \right| \\ &\leq 2|D\mathcal{H}(y)| \left| D\mathcal{H}(y+z) - D\mathcal{H}(y) \right| \\ &\quad + 2|D\mathcal{H}(y+z)| \left| \frac{\mathcal{H}(y+z) - \mathcal{H}(y) - D\mathcal{H}(y) \cdot z}{|z|} \right| \\ &\leq \|D\mathcal{H}\|_{L^\infty(\mathbb{B}_{2\varepsilon}(x))} \left( \omega_1(|z|) + \omega_2(|z|) \right), \end{aligned}$$

for some moduli of continuity  $\omega_1, \omega_2$  (by the  $C^1$ -regularity of  $\mathcal{H}$ ). Hence, (6.9) has been established. Further, since  $D(\mathcal{H}^2) \equiv 0$  on  $\mathcal{K}$  we obviously have

$$(6.10) \quad \max_{y \in \mathcal{K}} \left| \mathcal{H}^2(y+z) - \mathcal{H}^2(y) \right| = o(|z|)$$

as  $z \rightarrow 0$ , while also

$$(6.11) \quad \max_{y \in \mathcal{K}} \left| \mathcal{H}^2(z+y) - \mathcal{H}^2(y) - \left( D\mathcal{H}(y) \otimes D\mathcal{H}(y) \right) : z \otimes z \right| \leq o(|z|^2),$$

as  $z \rightarrow 0$ . The inequality (6.11) is an easy consequence of (6.9) and the identity

$$\mathcal{H}^2(w+y) - \mathcal{H}^2(y) = w \cdot \int_0^1 D(\mathcal{H}^2)(y + \lambda w) d\lambda.$$

Conclusively, by (6.9)-(6.11) the function  $\mathcal{H}^2$  is twice (Whitney) differentiable on the closed set  $\mathcal{K}$  with Whitney hessian

$$D^2\left(\frac{1}{2}\mathcal{H}^2\right) = D(H(D^2u)) \otimes D(H(D^2u)).$$

By the Whitney extension theorem ([W, M, F]), there exists an extension  $\Phi \in C^2(\mathbb{R}^n)$  such that, on  $\mathcal{K}$  we have  $\Phi = \mathcal{H}^2/2 \equiv 0$ ,  $D\Phi = D(\mathcal{H}^2/2) \equiv 0$  and

$$D^2\Phi = D^2(\mathcal{H}^2/2) = D(H(D^2u)) \otimes D(H(D^2u)).$$

Let now  $\zeta$  be the cut-off function of Claim 17. The test function  $\phi := \Phi\zeta$  satisfies  $\phi \in W_0^{2,\infty}(\Omega_\varepsilon(x)) \cap C^2(\overline{\Omega_\varepsilon(x)})$  and

$$D^2\phi(x) = D(H(D^2u))(x) \otimes D(H(D^2u))(x).$$

By inserting the above to (6.4) we conclude that  $A_\infty^2 u(x) = 0$  for the arbitrary point  $x \in \Omega$ .

(B) (4)  $\Rightarrow$  (3): We rewrite the equation  $A_\infty^2 u = 0$  as

$$\mathbf{H}'(A : D^2 u) \left[ A : D(\mathbf{H}(A : D^2 u)) \otimes D(\mathbf{H}(A : D^2 u)) \right] = 0$$

which by decomposing the positive matrix  $A$  as  $A = A^{1/2} A^{1/2}$ , we reformulate as

$$\mathbf{H}'(A : D^2 u) \left| A^{1/2} D(\mathbf{H}(A : D^2 u)) \right|^2 = 0.$$

By the PDE it follows that on the open set  $\Omega^* := \{\mathbf{H}'(A : D^2 u) \neq 0\}$  we have  $D(\mathbf{H}(A : D^2 u)) = 0$  because  $A^{1/2}$  is invertible. On the other hand, on  $\Omega \setminus \Omega^*$  we have  $\mathbf{H}'(A : D^2 u) = 0$  which gives  $D(\mathbf{H}(A : D^2 u)) = \mathbf{H}'(A : D^2 u) D(A : D^2 u) = 0$ . Consequently,  $D(\mathbf{H}(A : D^2 u)) = 0$  on the whole of  $\Omega$  and by the connectivity of the domain we infer that  $\mathbf{H}(A : D^2 u) \equiv c$  for some  $c \geq 0$ .

(3)  $\Rightarrow$  (2): If  $\mathbf{H}(A : D^2 u) \equiv c$  on  $\Omega$ , since  $\{\mathbf{H} = c\}$  consists of at most 2 points and  $A : D^2 u(\Omega')$  is a connected set, we obtain  $A : D^2 u(\Omega') \subseteq \{C\}$  where  $C \in \{\mathbf{H} = c\}$ .

(2)  $\Rightarrow$  (1): Let us denote the  $n$ -Lebesgue and the  $n - 1$ -Hausdorff measure as in [EG] by  $\mathcal{L}^n$  and  $\mathcal{H}^{n-1}$  respectively. Fix  $\Omega' \Subset \Omega$  and  $\phi \in W_0^{2,\infty}(\Omega')$ . Extend  $\phi$  by zero on  $\mathbb{R}^n \setminus \Omega'$ , consider the standard mollification  $\eta^\varepsilon * D^2 \phi$  of it by convolution and let  $\Omega'_\delta$  be a piecewise smooth domain containing the support of  $\eta^\varepsilon * D^2 \phi$  (e.g. union of finite many balls). By the Gauss-Green theorem, we have

$$\int_{\Omega'_\delta} \eta^\varepsilon * D^2 \phi d\mathcal{L}^n = \int_{\Omega'_\delta} D(\eta^\varepsilon * D\phi) d\mathcal{L}^n = \int_{\partial\Omega'_\delta} (\eta^\varepsilon * D\phi) \otimes \nu d\mathcal{H}^{n-1} = 0$$

and by letting  $\varepsilon \rightarrow 0$  and  $\delta \rightarrow 0$ , we get that the average of  $D^2 \phi$  over  $\Omega'$  vanishes. Hence, since  $A : D^2 u \equiv C$  on  $\Omega$ , we obtain

$$C = \int_{\Omega'} A : D^2 u d\mathcal{L}^n = \int_{\Omega'} (A : D^2 u + A : D^2 \phi) d\mathcal{L}^n$$

By applying  $\mathbf{H}$  to the above equality, Jensen's inequality for level-convex functions (see e.g. [BJW1, BJW2]) implies

$$\begin{aligned} \operatorname{ess\,sup}_{\Omega'} \mathbf{H}(A : D^2 u) &= \mathbf{H}(C) = \mathbf{H} \left( \int_{\Omega'} (A : D^2 u + A : D^2 \phi) d\mathcal{L}^n \right) \\ &\leq \operatorname{ess\,sup}_{\Omega'} \mathbf{H}(A : D^2 u + A : D^2 \phi) \end{aligned}$$

which yields that  $u \in \operatorname{AM}^2(E_\infty, \Omega)$ .

(1)  $\Rightarrow$  (4): Proved in part (A) above. The proof of Theorem 14 is now complete.  $\square$

**Proof of Corollary 15.** In view of the equivalences among (1)-(4) of Theorem 14(B), it follows that either the Dirichlet problem for the PDE or for 2nd order Absolute Minimisers reduces to the uniqueness of solution in  $C^3(\Omega) \cap W_g^{2,\infty}(\Omega)$  to the Dirichlet problem for the linear elliptic 2nd order PDE

$$\begin{cases} A : D^2 u = C, & \text{in } \Omega, \\ u = g, Du = Dg, & \text{on } \partial\Omega, \end{cases}$$

for some  $C \in \mathbb{R}$ , which is over-determined and has at most 1 solution.  $\square$

## 7. EXISTENCE OF $\mathcal{D}$ -SOLUTIONS TO THE DIRICHLET PROBLEM FOR $A_\infty^2$

Herein we establish the existence of  $\mathcal{D}$ -solutions with extra properties to the Dirichlet problem for (1.2). These solutions are in a sense “critical points” of (1.1) and generally non-minimising and non-unique. They are obtained *without imposing any kind of convexity, neither level-convexity nor quasiconvexity nor “BJW-convexity” (the notion of  $L^\infty$ -quasiconvexity of [BJW1])*. Actually, our *only assumption on  $H$  is that it is  $C^1$  and depends on  $X$  via  $X^2 = X^\top X$* .

The method we employ has two main steps. First, given  $\Omega \subseteq \mathbb{R}^n$  open and  $g \in W^{2,\infty}(\Omega)$ , we solve the fully nonlinear PDE

$$(7.1) \quad H(D^2u) = C, \quad \text{a.e. on } \Omega,$$

for admissible large enough “energy level”  $C > 0$  depending on the data  $g$ . For this we use the celebrated *Baire Category method* of Dacorogna-Marcellini (see [DM, D]) which is a convenient analytic alternative to Gromov’s Convex Integration. Next, we use the machinery of  $\mathcal{D}$ -solutions to make the next *non-rigorous statement* precise: every solution  $u$  to (7.1) solves (1.2) because  $D(H(D^2u)) \equiv 0$  and (1.2) “equals” (1.3). This is indeed true in the class of classical/strong solutions in  $C^3(\Omega)$  or  $W^{3,\infty}(\Omega)$ , but not in the natural  $W^{2,\infty}(\Omega)$  class. This method of constructing critical point solutions has previously been applied successfully to the vector-valued first order case and its generalisations, see [K8, K9, AK, CKP]. The principal result of this section therefore is:

**Theorem 18** (Existence of  $\mathcal{D}$ -solutions to the Dirichlet problem for  $A_\infty^2 u = 0$ ). *Let  $H : \mathbb{R}_s^{n \otimes 2} \rightarrow \mathbb{R}$  be such that  $H(X) = h(X^2)$  for some  $h \in C^1(\mathbb{R}_s^{n \otimes 2})$ . Consider also an open set  $\Omega \subseteq \mathbb{R}^n$  and fix  $g \in W^{2,\infty}(\Omega)$ . Then, for any “energy level”  $c > \|D^2g\|_{L^\infty(\Omega)}$ , the Dirichlet problem for (1.2)*

$$(7.2) \quad \begin{cases} A_\infty^2 u = 0, & \text{in } \Omega, \\ u = g, \quad Du = Dg, & \text{on } \partial\Omega, \end{cases}$$

*has (an infinite set of)  $\mathcal{D}$ -solutions in the class*

$$\mathcal{A}_c := \left\{ v \in W_g^{2,\infty}(\Omega) : H(D^2v) = H(cI) \text{ a.e. on } \Omega \right\}.$$

*Namely, there is a set of  $u$ ’s in  $\mathcal{A}_c$  such that (in view of Definitions 2 and 3)*

$$(7.3) \quad \int_{\overline{\mathbb{R}}_s^{n \otimes 3}} \Phi(\mathbf{X}) \left[ H_X(D^2u(x))^{\otimes 3} : \mathbf{X}^{\otimes 2} \right] d[\mathcal{D}^3u(x)](\mathbf{X}) = 0, \quad \text{a.e. } x \in \Omega,$$

*for any diffuse 3rd derivative  $\mathcal{D}^3u \in \mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3})$  and any  $\Phi \in C_c(\mathbb{R}_s^{n \otimes 3})$ .*

**Proof of Theorem 18.** Let  $\Omega \subseteq \mathbb{R}^n$  be a given open set and  $g \in W^{2,\infty}(\Omega)$ ,  $n \in \mathbb{N}$ . We begin by showing the next result.

**Claim 19.** *For any fixed  $c > \|D^2g\|_{L^\infty(\Omega)}$ , there exist (an infinite set of) solutions in  $W_g^{2,\infty}(\Omega)$  such that*

$$\begin{cases} D^2u^\top D^2u = c^2I, & \text{a.e. in } \Omega, \\ u = g, \quad Du = Dg, & \text{on } \partial\Omega. \end{cases}$$

**Proof of Claim 19.** Let  $\{\lambda_1(X), \dots, \lambda_n(X)\}$  symbolise the eigenvalues of the symmetric matrix  $X \in \mathbb{R}_s^{n \otimes 2}$  in increasing order. By the results of [[DM], p. 200] the Dirichlet problem

$$\begin{cases} |\lambda_i(D^2 v)| = 1, & \text{a.e. on } \Omega, \alpha = 1, \dots, n, \\ v = g/c, & \text{on } \partial\Omega, \end{cases}$$

has solutions  $v \in W_{g/c}^{2,\infty}(\Omega, \mathbb{R}^N)$  because we have that

$$\begin{aligned} \max_{i=1,\dots,n} \left\{ \operatorname{ess\,sup}_\Omega |\lambda_i(D^2(g/c))| \right\} &\leq \frac{1}{c} \operatorname{ess\,sup}_\Omega \left\{ \max_{|e|=1} |D^2 g : (e \otimes e)| \right\} \\ &\leq \frac{1}{c} \|D^2 g\|_{L^\infty(\Omega)} < 1, \end{aligned}$$

a.e. on  $\Omega$ . By rescaling as  $u := cv$ , we get existence of solutions  $u \in W_g^{2,\infty}(\Omega)$  to

$$\begin{cases} |\lambda_i(D^2 u)| = c, & \text{a.e. on } \Omega, \alpha = 1, \dots, n, \\ v = g, & \text{on } \partial\Omega. \end{cases}$$

Note now that  $\lambda_i(D^2 u)^2 = \lambda_i(D^2 u^\top D^2 u)$  for all  $i = 1, \dots, n$ . Hence, by the Spectral Theorem there is an  $L^\infty$  map with values in the orthogonal matrices  $O : \Omega \subseteq \mathbb{R}^n \rightarrow O(n, \mathbb{R}) \subseteq \mathbb{R}^{n \otimes 2}$  such that

$$D^2 u^\top D^2 u = O \begin{bmatrix} \lambda_1(D^2 u)^2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n(D^2 u)^2 \end{bmatrix} O^\top = O(c^2 \mathbf{I}) O^\top = c^2 \mathbf{I},$$

a.e. on  $\Omega$ . The claim thus ensues.  $\square$

Now we complete the proof of the theorem. By our assumption on  $H$ , for any  $u \in W_g^{2,\infty}(\Omega)$  as in Claim 19 we have

$$(7.4) \quad H(D^2 u) = h(D^2 u^\top D^2 u) = h(c^2 \mathbf{I}) = h((c\mathbf{I})^2) = H(c\mathbf{I}),$$

a.e. on  $\Omega$ . Hence,  $u \in \mathcal{A}_c$ . Note also that by (7.4) we have  $H(D^2 u(x)) = \text{const}$  for a.e.  $x \in \Omega$ . The next claim completes the proof.

**Claim 20.** *If  $H(D^2 u) = C$  a.e. on  $\Omega$ , then  $u$  is a  $\mathcal{D}$ -solution to (1.2), that is (7.3) holds for any fixed  $\Phi \in C_c(\mathbb{R}_s^{n \otimes 3})$  and a.e.  $x \in \Omega$  for any diffuse 3rd derivative of  $u$*

$$(7.5) \quad \delta_{D^{1,h_m} D^2 u} \xrightarrow{*} \mathcal{D}^3 u \quad \text{in } \mathcal{Y}(\Omega, \overline{\mathbb{R}}_s^{n \otimes 3}), \quad \text{as } m \rightarrow \infty.$$

**Proof of Claim 20.** Fix such an  $x \in \Omega$ ,  $0 < |h| < \text{dist}(x, \partial\Omega)$  and  $k \in \{1, \dots, n\}$ . By Taylor's theorem, we have

$$\begin{aligned} 0 &= H(D^2 u(x + he^k)) - H(D^2 u(x)) \\ &= \sum_{i,j} \int_0^1 H_{X_{ij}} \left( D^2 u(x) + \lambda [D^2 u(x + he^k) - D^2 u(x)] \right) d\lambda \\ &\quad \cdot [D_{ij}^2 u(x + he^k) - D_{ij}^2 u(x)]. \end{aligned}$$

This implies for any  $k = 1, \dots, n$  the identity

$$\sum_{i,j} H_{X_{ij}}(D^2 u(x)) (D_k^{1,h} D_{ij}^2 u)(x) =$$



$$\begin{aligned}
&= \sum_{i,j} \left\{ \int_0^1 \left[ -H_{X_{ij}} \left( D^2 u(x) + \lambda \left[ D^2 u(x + h e^k) - D^2 u(x) \right] \right) \right. \right. \\
(7.6) \quad &\quad \left. \left. + H_{X_{ij}} \left( D^2 u(x) \right) \right] d\lambda \right\} (D_k^{1,h} D_{ij}^2 u)(x) \\
&=: \sum_{ij} \mathcal{E}_{ijk}(x, h) (D_k^{1,h} D_{ij}^2 u)(x)
\end{aligned}$$

where  $\mathcal{E}_{ijk}$  is the “error tensor”. By taking (7.6) for  $k = p, q$ , multiplying these two equations with  $H_{X_{pq}}(D^2 u(x))$  and summing in  $p, q \in \{1, \dots, n\}$ , we obtain

$$\begin{aligned}
&\sum_{i,j,r,s,p,q} \left[ \left( H_{X_{pq}}(D^2 u) H_{X_{ij}}(D^2 u) H_{X_{rs}}(D^2 u) \right) (D_p^{1,h} D_{ij}^2 u) (D_q^{1,h} D_{rs}^2 u) \right] \\
(7.7) \quad &= \sum_{i,j,r,s,p,q} \left\{ H_{X_{pq}}(D^2 u) \mathcal{E}_{ijp}(\cdot, h) \mathcal{E}_{rsq}(\cdot, h) \right\} (D_p^{1,h} D_{ij}^2 u) (D_q^{1,h} D_{rs}^2 u) \\
&=: \sum_{i,j,r,s,p,q} \mathcal{E}_{pqijrs}(\cdot, h) (D_p^{1,h} D_{ij}^2 u) (D_q^{1,h} D_{rs}^2 u).
\end{aligned}$$

Let  $(h_m)_1^\infty$  be an infinitesimal sequence giving rise to a diffuse 3rd derivative as in (7.5). We rewrite (7.7) for  $h = h_m$  compactly as

$$\left( (H_X(D^2 u))^{\otimes 3} - \mathcal{E}(\cdot, h_m) \right) : (D^{1,h_m} D^2 u)^{\otimes 2} = 0$$

for  $m \in \mathbb{N}$ . Then for any  $\Phi \in C_c(\mathbb{R}_s^{n^{\otimes 3}})$  and  $\phi \in C_c(\Omega)$ , this yields

$$(7.8) \quad \int_\Omega \phi \int_{\mathbb{R}_s^{n^{\otimes 3}}} \Phi(\mathbf{X}) \left[ \left( (H_X(D^2 u))^{\otimes 3} - \mathcal{E}(\cdot, h_m) \right) : \mathbf{X}^{\otimes 2} \right] d[\delta_{D^{1,h_m} D^2 u}](\mathbf{X}) = 0.$$

Since  $|D^2 u| \in L^\infty(\Omega)$ , by the continuity of the translation operation in  $L^1$  we have  $|D^2 u(\cdot + z) - D^2 u| \rightarrow 0$  as  $z \rightarrow 0$ , in  $L_{\text{loc}}^1(\Omega)$ . Hence, along perhaps a further subsequence  $(m_i)_{i=1}^\infty$  we have  $D^2 u(x + h_m e^k) \rightarrow D^2 u(x)$  for a.e.  $x \in \Omega$  as  $m \rightarrow \infty$ ,  $k = 1, \dots, n$ . Since  $H \in C^1(\mathbb{R}_s^{n^{\otimes 2}})$  and  $|D^2 u| \in L^\infty(\Omega)$ , the Dominated Convergence theorem and the definition of the errors in (7.6)-(7.7) imply that  $|\mathcal{E}(\cdot, h_m)| \rightarrow 0$  in  $L_{\text{loc}}^1(\Omega)$  subsequentially as  $m \rightarrow \infty$ . We define the Carathéodory functions

$$\begin{aligned}
\Psi^m(x, \mathbf{X}) &:= \phi(x) \Phi(\mathbf{X}) \left| \left( (H_X(D^2 u))^{\otimes 3} - \mathcal{E}(\cdot, h_m) \right) : \mathbf{X}^{\otimes 2} \right|, \\
\Psi^\infty(x, \mathbf{X}) &:= \phi(x) \Phi(\mathbf{X}) \left| (H_X(D^2 u))^{\otimes 3} : \mathbf{X}^{\otimes 2} \right|
\end{aligned}$$

which are elements of the Banach space  $L^1(E, C(\overline{\mathbb{R}_s^{n^{\otimes 3}}}))$  (because of the compactness of the supports of  $\phi, \Phi$ ) and we also have

$$(7.9) \quad \Psi^m \rightarrow \Psi, \text{ as } m \rightarrow \infty \text{ in } L^1(E, C(\overline{\mathbb{R}_s^{n^{\otimes 3}}}))$$

which is a consequence of that  $|\mathcal{E}(\cdot, h_m)| \rightarrow 0$  in  $L_{\text{loc}}^1(\Omega)$  and the estimate

$$\|\Psi^{m_k} - \Psi^\infty\|_{L^1(E, C(\overline{\mathbb{R}_s^{n^{\otimes 3}}}))} \leq \max_{\mathbf{X} \in \text{supp}(\Phi)} |\Phi(\mathbf{X})| |\mathbf{X}|^2 \int_{\text{supp}(\phi)} |\phi \mathcal{E}(\cdot, h_m)|.$$

The weak\*-strong continuity of the duality pairing (2.1), (7.9) and (7.5) allow us to pass to the limit in (7.8) as  $m \rightarrow \infty$  and deduce

$$\int_{\Omega} \phi \int_{\mathbb{R}_s^{\otimes 3}} \Phi(\mathbf{X}) \left[ (\mathbf{H}_X(\mathbf{D}^2 u))^{\otimes 3} : \mathbf{X}^{\otimes 2} \right] d[\mathcal{D}^3 u](\mathbf{X}) = 0.$$

Since  $\phi \in C_c(\Omega)$  is arbitrary, (7.3) follows and the claim ensues.  $\square$

The proof of the theorem is now complete.  $\square$

## 8. EXPLICIT $p$ -BIHARMONIC AND $\infty$ -BIHARMONIC FUNCTIONS IN $1D$

In this section we give explicit solutions to the Dirichlet problem for the  $p$ -Bilaplacian and the  $\infty$ -Bilaplacian when  $n = 1$ . In this case the equations are

$$(8.1) \quad \Delta_p^2 u = (|u''|^{p-2} u'')'' = 0$$

$$(8.2) \quad \Delta_\infty^2 u = (u'')^3 (u''')^2 = 0$$

The weak solutions we construct for (8.1) are obtained by solving the equation explicitly for even exponents  $p \in 2\mathbb{N}$ , while the  $\mathcal{D}$ -solutions we construct for (8.2) are piecewise quadratic. In either case the solutions in general have at least 1 singular point in their domain, unless the boundary data can be interpolated by a quadratic polynomial function in which case the solutions are smooth. Accordingly, the main result of this section is:

**Theorem 21** (Explicit generalised solutions). *Consider  $a, b, A, B, A', B' \in \mathbb{R}$  with  $a < b$  and set*

$$E := \frac{B' - A'}{b - a} - \frac{2(B - A - A'(b - a))}{(b - a)^2}.$$

(A) [ $p$ -Bilaplacian] *Let  $p \in 2\mathbb{N}$ . Then, the problem*

$$(8.3) \quad \begin{cases} \Delta_p^2 u = 0, & \text{in } (a, b) \subseteq \mathbb{R}, \\ u(a) = A, & u(b) = B, \\ u'(a) = A', & u'(b) = B', \end{cases}$$

*has a unique weak solution  $u_p \in W^{2,p}(a, b)$  given by:*

(i) *For critical data satisfying  $E = 0$  (which can be interpolated by a quadratic function),*

$$(8.4) \quad u_p(x) = A + A'(x - a) + \frac{1}{2} \left( \frac{B' - A'}{b - a} \right) (x - a)^2.$$

(ii) *For non-critical data satisfying  $E \neq 0$  (which can not be interpolated by a quadratic function),*

$$(8.5) \quad u_p(x) = A + \left( A' - \frac{p-1}{p\lambda} |\lambda a + \mu|^{\frac{p}{p-1}} \right) (x - a) + \frac{p-1}{p\lambda} \int_a^x |\lambda t + \mu|^{\frac{p}{p-1}} dt.$$

*Further,  $(\lambda, \mu) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$  is the unique solution to the algebraic equations:*

$$(8.6) \quad \begin{cases} (B' - A') \frac{p\lambda}{p-1} = |\lambda b + \mu|^{\frac{p}{p-1}} - |\lambda a + \mu|^{\frac{p}{p-1}}, \\ (B - A - A'(b - a)) \frac{p\lambda}{p-1} = \int_a^b |\lambda t + \mu|^{\frac{p}{p-1}} dt - |\lambda a + \mu|^{\frac{p}{p-1}} (b - a). \end{cases}$$

*In particular,  $u_p \in C^\infty((a, b) \setminus \{-\mu/\lambda\})$ .*

(B) [ $\infty$ -Bilaplacian] For any large enough “energy level”  $C > 0$  depending only on the boundary data (see (8.10)), the problem (8.3) for  $p = \infty$  has a unique piecewise quadratic  $\mathcal{D}$ -solution  $u_\infty \in W^{2,\infty}(a, b)$  given by

$$(8.7) \quad u_\infty(x) = A + A'(x - a) + C \int_a^x \left[ \mathcal{L}^1([a, t] \cap I_C) - \mathcal{L}^1([a, t] \setminus I_C) \right] dt.$$

Here  $I_C = [x_C, y_C]$  is the interval with endpoints

$$(8.8) \quad x_C = \frac{-K - L^2 + 2bL}{2L}, \quad y_C = \frac{-K + L^2 + 2bL}{2L}$$

where

$$(8.9) \quad K = \frac{2(B - A - A'(b - a)) + C(b - a)^2}{2C}, \quad L = \frac{B' - A' + C(b - a)}{2C}.$$

In particular,  $u_\infty$  satisfies  $|u_\infty''| = C$  a.e. on  $\Omega$  and  $u_\infty \in C^\infty((a, b) \setminus \{x_C, y_C\})$ .

**Remark 22.** We note that the solution  $u_\infty$  above is not the limit of  $u_p$  as  $p \rightarrow \infty$ . The function  $\lim_p u_p$  is indeed Absolutely Minimising by the results of Section 5 but we do not prove here that it solves in the  $\mathcal{D}$ -sense the equation. Instead, we solve (8.2) by solving the fully nonlinear equation  $|u''| = C$  for a fixed energy level  $C > 0$  and using the previous section to characterise it as a  $\mathcal{D}$ -solution to (8.2). The numerics of the next section show that  $\lim_p u_p$  has at most 1 “breaking point” for the 2nd derivative in the domain of definition, while these solutions are “critical points” and as such have instead less regularity and 2 “breaking points” of their 2nd derivative.

**Proof of Theorem 21.** (A) Let  $u_p$  be a weak solution to (8.3). By standard convexity and variational arguments (see e.g. [E, D]), the solution exists and it is energy minimising and unique. Note now that the function  $\mathbb{R} \ni t \mapsto |t|^{p-2}t = t^{p-1} \in \mathbb{R}$  and its inverse  $t \mapsto t^{1/(p-1)}$  are odd because  $p \in 2\mathbb{N}$ . We obtain (i)-(ii) directly by differentiating twice the explicit formulas (8.4)-(8.5). By the previous observation, in either case this gives  $u_p''(x) = (\lambda x + \mu)^{\frac{1}{p-1}}$  for  $a < x < b$ , where in the case of (i) we have  $\mu = 0$  and  $\lambda = ((B' - A')/(b - a))^{p-1}$ , whilst in the case of (ii) the parameters  $(\lambda, \mu)$  are given (8.6). The latter is just a compatibility condition arising by the boundary conditions. In both cases we get that the function  $|u_p''|^{p-2}u_p''$  is affine and  $(|u_p''|^{p-2}u_p'')(x) = \lambda x + \mu$  for  $a < x < b$ . As a consequence,  $\Delta_p^2 u_p = 0$  weakly on  $(a, b)$ .

(B) Let  $x_C, y_C$  be given by (8.8)-(8.9) for any  $C > C^*$ ,  $C^*$  large enough to be specified. The compatibility requirement  $a \leq x_C < y_C \leq b$  (in order to have endpoints of an interval  $I_C = [x_C, y_C]$  which lies inside  $[a, b]$ ) after a calculation is equivalent to

$$(8.10) \quad \begin{cases} \frac{1}{C} \left[ \frac{(B' - A')^2}{4C} + \frac{(B' - A')(b - a)}{2} - (B - A - A'(b - a)) \right] \leq \frac{(b - a)^2}{4}, \\ \frac{1}{C} \left[ \frac{(B' - A')^2}{4C} - \frac{(B' - A')(b - a)}{2} \right] \leq \frac{(b - a)^2}{4}, \end{cases}$$

and hence for  $C^*$  large enough we indeed have existence. Thus,  $u_\infty$  as given by (8.7) is a well-defined  $W^{2,\infty}(a, b)$  function. By differentiating (8.7), we get  $u_\infty'' = C$  on  $(x_C, y_C)$  and  $u_\infty'' = -C$  on  $(a, x_C) \cup (y_C, b)$ , giving  $|u_\infty''| = C$  on  $(a, b) \setminus \{x_C, y_C\}$ . By Claim 20 of Section 7,  $u_\infty$  is a  $\mathcal{D}$ -solution to  $\Delta_\infty^2 u = 0$ . Note now that  $x_C, y_C$

satisfy the identities  $(x_C)^2 - (y_C)^2 + 2b(y_C - x_C) = K$  and  $y_C - x_C = L$  and by a calculation, it can be verified that the algebraic equations above are equivalent to

$$\begin{cases} B = A + A'(b-a) + C \int_a^b [\mathcal{L}^1([a, t] \cap I_C) - \mathcal{L}^1([a, t] \setminus I_C)] dt, \\ B' = A' + C [2\mathcal{L}^1(I_C) - (b-a)]. \end{cases}$$

The latter pair of equations are just a restatement of the fact that  $u_\infty$  satisfies the boundary conditions. The theorem ensues.  $\square$

## 9. NUMERICAL APPROXIMATIONS OF $\infty$ -BIHARMONIC FUNCTIONS

In this section we illustrate some of the properties of  $\infty$ -Biharmonic functions using numerical techniques. We present results from a numerical scheme that makes use of a  $p$ -Biharmonic approximation, that is, we make use of the derivation through the  $p$ -limiting process given in Section 3. Our numerical scheme of choice is a finite element method and is fully described in [KP2] where we prove for fixed  $p$  that the scheme converges to the weak solution of the  $p$ -Biharmonic problem. The results there illustrate that for practical purposes, as one would expect, the approximation of  $p$ -Biharmonic functions for large  $p$  gives good resolution of candidate  $\infty$ -Biharmonic functions. In this work for brevity we restrict ourselves to presenting only some results.

**Test 1: the 1-dimensional problem.** We consider the Dirichlet problem (8.3) for the  $p$ -Bilaplacian (8.1) for  $n = 1$  with the data  $A, B, A', B'$  being given by the values of the cubic function

$$(9.1) \quad g(x) = \frac{1}{120}(4x-3)(2x-1)(4x-1)$$

on  $(0, 1)$ . We simulate the  $p$ -Bilaplacian (8.1) for increasing values of  $p$  and present the results in Figure 1 indicating that in the limit the  $\infty$ -Biharmonic function should be piecewise quadratic.

**Test 2: the 2-dimensional problem.** Now we illustrate some of the complicated behaviour of the  $p$ -Bilaplacian for  $n = 2$ :

$$(9.2) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = 0, & \text{in } \Omega = [-1, 1]^2, \\ u = g, \quad Du = Dg, & \text{on } \partial\Omega, \end{cases}$$

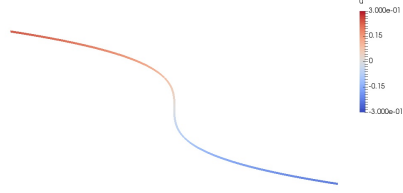
where  $g$  is prescribed as

$$(9.3) \quad g(x, y) = \frac{1}{20} \cos(\pi x) \cos(\pi y).$$

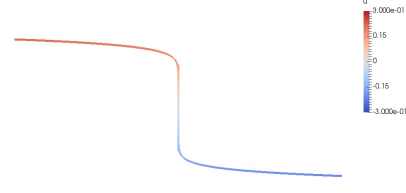
We simulate the  $p$ -Bilaplacian for increasing values of  $p$  and present the results in Figure 2 indicating that in the limit the  $\infty$ -Biharmonic function should be piecewise quadratic however the behaviour is quite unexpected and complicated interface patterns emerge even with this relatively simple boundary data.

**Acknowledgement.** N.K. would like to thank Craig Evans, Robert Jensen, Roger Moser, Juan Manfredi and Jan Kristensen for their inspiring mathematical discussions and especially their illuminating remarks on  $\mathcal{D}$ -solutions and on 2nd order  $L^\infty$  variational problems.

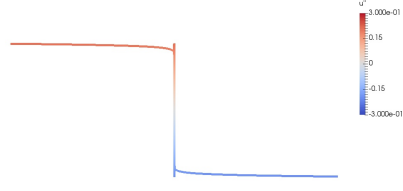
FIGURE 1. A mixed finite element approximations to an  $\infty$ -Biharmonic function using  $p$ -Biharmonic functions for various  $p$  for the problem given by (8.3) and (9.1). Notice that as  $p$  increases,  $u''$  tends to a piecewise constant up to Gibbs oscillations. This is an indication the solution is indeed piecewise quadratic.



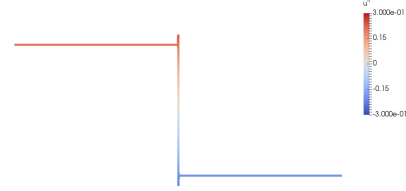
(a) The approximation to  $u''$ , the Laplacian of the solution of the 4-Bilaplacian.



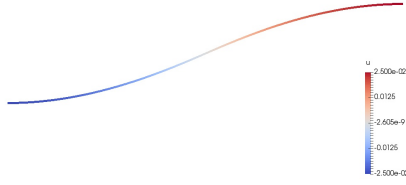
(b) The approximation to  $u''$ , the Laplacian of the solution of the 12-Bilaplacian.



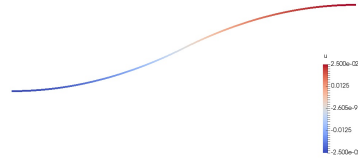
(c) The approximation to  $u''$ , the Laplacian of the solution of the 42-Bilaplacian.



(d) The approximation to  $u''$ , the Laplacian of the solution of the 202-Bilaplacian.

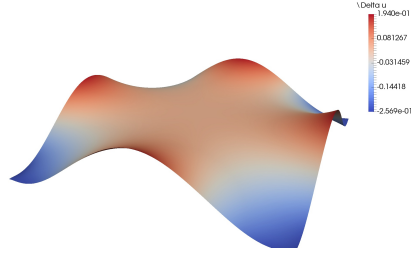


(e) The approximation to  $u$ , the solution of the 4-Bilaplacian.

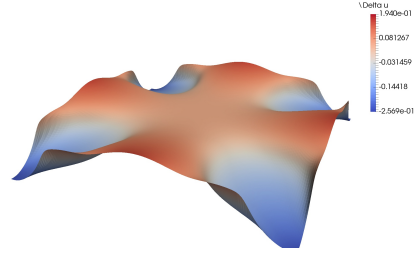


(f) The approximation to  $u$ , the solution of the 202-Bilaplacian.

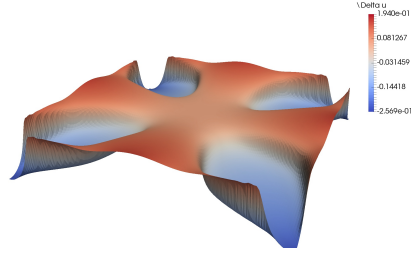
FIGURE 2. A mixed finite element approximations to an  $\infty$ -Biharmonic function using  $p$ -Biharmonic functions for various  $p$  for the problem given by (9.2) and (9.3). Notice that as  $p$  increases,  $\Delta u$  tends to be piecewise constant. This is an indication the solution satisfies the Poisson equation with piecewise constant right hand side albeit with an extremely complicated solution pattern that clearly warrants further investigation.



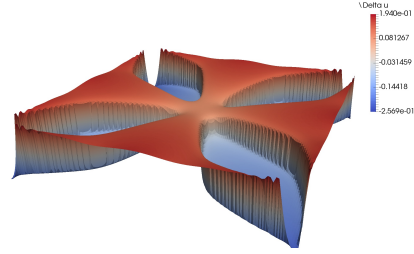
(a) The approximation to  $\Delta u$ , the Laplacian of the solution of the 4-Bilaplacian.



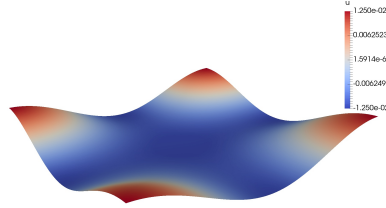
(b) The approximation to  $\Delta u$ , the Laplacian of the solution of the 42-Bilaplacian.



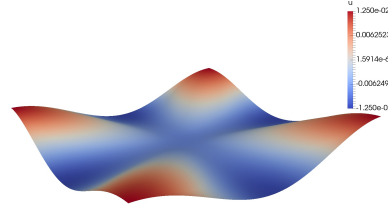
(c) The approximation to  $\Delta u$ , the Laplacian of the solution of the 68-Bilaplacian.



(d) The approximation to  $\Delta u$ , the Laplacian of the solution of the 142-Bilaplacian.



(e) The approximation to  $u$ , the solution of the 4-Bilaplacian.



(f) The approximation to  $u$ , the solution of the 142-Bilaplacian.

## REFERENCES

- AK. H. Abugirda, N. Katzourakis, *Existence of 1D vectorial Absolute Minimisers in  $L^\infty$  under minimal assumptions*, Proceedings of the AMS 145:6, 2567-2575 (2017).
1. H. Abugirda and B. Ayanbayev, N. Katzourakis, *Rigidity and flatness of the image of certain classes of mappings having tangential Laplacian*, <https://arxiv.org/pdf/1704.04492.pdf>.
  2. B. Ayanbayev, N. Katzourakis, *A Pointwise Characterisation of the PDE system of vectorial Calculus of variations in  $L^\infty$* , Proc. Royal Soc. Edinburgh A, in press.

- AM. L. Ambrosio, J. Malý, *Very weak notions of differentiability*, Proceedings of the Royal Society of Edinburgh A 137 (2007), 447-455.
- A1. G. Aronsson, *Minimization problems for the functional  $\sup_x \mathcal{F}(x, f(x), f'(x))$* , Arkiv für Mat. 6 (1965), 33-53.
- A2. G. Aronsson, *Minimization problems for the functional  $\sup_x \mathcal{F}(x, f(x), f'(x))$  II*, Arkiv für Mat. 6 (1966), 409-431.
- A3. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551-561.
- A4. G. Aronsson, *On the partial differential equation  $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$* , Arkiv für Mat. 7 (1968), 395-425.
- A5. G. Aronsson, *Minimization problems for the functional  $\sup_x \mathcal{F}(x, f(x), f'(x))$  III*, Arkiv für Mat. (1969), 509-512.
- A6. G. Aronsson, *On Certain Singular Solutions of the Partial Differential Equation  $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$* , Manuscripta Math. 47 (1984), no 1-3, 133-151.
- A7. G. Aronsson, *Construction of Singular Solutions to the  $p$ -Harmonic Equation and its Limit Equation for  $p = \infty$* , Manuscripta Math. 56 (1986), 135-158.
- AB. G. Aronsson, E.N. Barron,  *$L^\infty$  Variational Problems with Running Costs and Constraints*, Appl Math Optim 65, 53-90 (2012).
- BJW1. E. N. Barron, R. Jensen and C. Wang, *The Euler equation and absolute minimizers of  $L^\infty$  functionals*, Arch. Rational Mech. Analysis 157 (2001), 255-283.
- BJW2. E. N. Barron, R. Jensen, C. Wang, *Lower Semicontinuity of  $L^\infty$  Functionals* Ann. I. H. Poincaré AN 18, 4 (2001) 495-517.
- CFV. C. Castaing, P. R. de Fitte, M. Valadier, *Young Measures on Topological spaces with Applications in Control Theory and Probability Theory*, Mathematics and Its Applications, Kluwer Academic Publishers, 2004.
- C. M. G. Crandall, *A visit with the  $\infty$ -Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes 1927, CIME, Cetraro Italy 2005.
- CKP. G. Croce, N. Katzourakis, G. Pisante,  *$\mathcal{D}$ -solutions to the system of vectorial Calculus of Variations in  $L^\infty$  via the singular value problem*, Discrete and Continuous Dynamical Systems 37:12, 6165-6181 (2017).
- D. B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2008.
- DM. B. Dacorogna, P. Marcellini, *Implicit Partial Differential Equations*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 1999.
- Da. J.M. Danskin, *The theory of min-max with application*, SIAM Journal on Applied Mathematics, 14 (1966), 641-664.
- DS. C. De Lellis, L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Annals of Mathematics 170, 1417-1436 (2009).
- Ed. R.E. Edwards, *Functional Analysis: Theory and Applications*, Dover, 2011.
- E. L.C. Evans, *Partial Differential Equations*, AMS Graduate Studies in Mathematics 19.1, 2nd edition 2010.
- EG. L.C. Evans, R. Gariepy, *Measure theory and fine properties of functions*, Studies in advanced mathematics, CRC press, 1992.
- F. C. Fefferman, *A sharp form of Whitney's extension theorem*, Annals of Mathematics, 161 (2005), 509-577.
- FG. L.C. Florescu, C. Godet-Thobie, *Young measures and compactness in metric spaces*, De Gruyter, 2012.
- FL. I. Fonseca, G. Leoni, *Modern methods in the Calculus of Variations:  $L^p$  spaces*, Springer Monographs in Mathematics, 2007.
- GS. A. Gastel, C. Scheven, *Regularity of polyharmonic maps in the critical dimension*, Communications in analysis and geometry 17 (2), 185-226, 2009.
- GM. M. Giaquinta, L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Publications of the Scuola Normale Superiore 11, Springer, 2012.
- GT. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, reprint of the 1998 edition, Springer.
- HM. P. Hornung, R. Moser, *Intrinsically  $p$ -Biharmonic maps*, Calculus of Variations and Partial Differential Equations, 51 (3-4), 597-620, 2011.

- K1. N. Katzourakis,  $L^\infty$ -Variational Problems for Maps and the Aronsson PDE system, J. Differential Equations, Volume 253, Issue 7 (2012), 2123-2139.
- K2. N. Katzourakis,  $\infty$ -Minimal Submanifolds, Proceedings of the AMS, 142 (2014) 2797-2811.
- K3. N. Katzourakis, On the Structure of  $\infty$ -Harmonic Maps, Communications in PDE, Volume 39, Issue 11 (2014), 2091-2124.
- K4. N. Katzourakis, Explicit 2D  $\infty$ -Harmonic Maps whose Interfaces have Junctions and Corners, Comptes Rendus Acad. Sci. Paris, Ser.I, 351 (2013) 677-680.
- K5. N. Katzourakis, Optimal  $\infty$ -Quasiconformal Immersions, ESAIM Control, Opt. and Calc. Var., Vol. 21, Number 2 (2015), 561-582.
- K6. N. Katzourakis, Nonuniqueness in Vector-valued Calculus of Variations in  $L^\infty$  and some Linear Elliptic Systems, Comm. on Pure and Appl. Anal., Vol. 14, 1, 313-327 (2015).
- K7. N. Katzourakis, An Introduction to viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in  $L^\infty$ , Springer Briefs in Mathematics, 2015.
- K8. N. Katzourakis, Absolutely minimising generalised solutions to the equations of vectorial Calculus of Variations in  $L^\infty$ , Calculus of Variations and PDE 56:1, 1-25 (2017).
- K9. N. Katzourakis, Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems, Journal of Differential Equations 23, 641-686 (2017).
- K10. N. Katzourakis, A new characterisation of  $\infty$ -Harmonic and  $p$ -Harmonic maps via affine variations in  $L^\infty$ , Electronic Journal of Differential Equations 2017:29, 1-19 (2017).
- K11. N. Katzourakis, Solutions of vectorial Hamilton-Jacobi equations are rank-one Absolute Minimisers in  $L^\infty$ , Advances in Nonlinear Analysis, in press (Published online: 2017-06-04).
- KP. N. Katzourakis, T. Pryer, On the numerical approximation of  $\infty$ -Harmonic mappings, Nonlinear Differential Equations and Applications 23:6, 1-23 (2016).
- KP2. N. Katzourakis, T. Pryer, On the numerical approximation of  $p$ -Biharmonic and  $\infty$ -Biharmonic functions, <https://arxiv.org/pdf/1701.07415.pdf>.
- KR. J. Kristensen, F. Rindler, Characterization of generalized gradient Young measures generated by sequences in  $W^{1,1}$  and  $BV$ , Arch. Rational Mech. Anal. 197, 539-598 (2010) and erratum Arch. Rational Mech. Anal. 203, 693-700 (2012).
- LP1. O. Lakkis, T. Pryer, A finite element method for nonlinear elliptic problems, SIAM Journal on Scientific Computing, 35(4):A2025-A2045, 2013.
- LP2. O. Lakkis, T. Pryer, An adaptive finite element method for the infinity laplacian, Numerical Mathematics and Advanced Applications, 283-291, 2013.
- M. B. Malgrange, Ideals of Differentiable Functions, Oxford Univ. Press, London, 1966.
- M. R. Moser, Regularity of minimizing extrinsic polyharmonic maps in the critical dimension, Manuscripta Mathematica, 131 (3-4), 475-485, 2010.
- MS. R. Moser, H. Schwetlick, Minimizers of a weighted maximum of the Gauss curvature, Annals of Global Analysis and Geometry, 41 (2), 199-207, 2012.
- Pe. P. Pedregal, Parametrized Measures and Variational Principles, Birkhäuser, 1997.
- PD. G. Pisante, B. Dacorogna, A general existence theorem for differential inclusions in the vector valued case, Portugaliae Mathematica, 2005.
- P. T. Pryer, On the finite element approximation of infinity-harmonic functions, Proc. of the Royal Soc. of Edinburgh Sec. A, to appear.
- S. Z. Sakellaris, Minimization of Scalar Curvature in Conformal Geometry, Ann. Global Anal. Geom. 51, 73-89 (2017).
- V. M. Valadier, Young measures, in "Methods of nonconvex analysis", Lecture Notes in Mathematics 1446, 152-188 (1990).
- W. H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934), 63-89.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF READING, WHITEKNIGHTS,  
PO BOX 220, READING RG6 6AX, UNITED KINGDOM

E-mail address: [n.katzourakis@reading.ac.uk](mailto:n.katzourakis@reading.ac.uk)

E-mail address: [t.pryer@reading.ac.uk](mailto:t.pryer@reading.ac.uk)